# AT

## 1 Convergence and continuity

**Theorem 1.1** (General principle of uniform convergence)**.** *Let $f_n : I \to \mathbb{R}$ be a sequence of functions. $f_n$ converges uniformly iff $f_n$ is uniformly Cauchy.*

*Proof.* $\Rightarrow$: Trivial. $\Leftarrow$: Prove pointwise convergence first. Use uniformly Cauchy property. Take limit. □

**Theorem 1.2.** *Let $f_n : I \to \mathbb{R}$ be a sequence of functions. $f_n \to f$ uniformly iff $\sup_{x \in I} |f_n(x) - f(x)| \to 0$ as $n \to 0$.*

*Proof.* Immediate from definition. □

**Theorem 1.3.** *Uniform limit of continuous function is continuous.*

*Proof.* $\epsilon/3$ argument. □

**Theorem 1.4.** *If $f_n \to f$ uniformly on $[a, b] \subseteq \mathbb{R}$ (closed bounded), and each $f_n$ is Riemann integrable on $[a, b]$, then $f$ is Riemann integrable and $\lim_n \int_a^b f_n = \int_a^b f$.*

*Proof.* First, note that each $f_n$ is bounded, so $f = f_n + (f - f_n$ is also bounded.

Let $\epsilon > 0$ be given. By unif. conv., there exists $N$ s.t. $\forall x \in [a, b]$, $|f_N(x) - f(x)| < \epsilon/(3(b - a))$. So, on any $I \subseteq [a, b]$, $\sup_I f \leq \sup_I f_N + \epsilon/(3(b - a))$.

By Riemann-integrability of $f_N$, there exists a dissection $\mathcal{D} = \{a = x_0 < x_1 < \ldots < x_k = b\}$ s.t.

$$U(f_N, \mathcal{D}) - L(f_N, \mathcal{D}) < \epsilon/3$$

Combine the results above, we get

$$U(f, \mathcal{D}) \leq U(f_N, \mathcal{D}) + \epsilon/3$$
$$L(f, \mathcal{D}) \geq L(f_N, \mathcal{D}) - \epsilon/3$$

Subtract, we get

$$U(f, \mathcal{D}) - L(f, \mathcal{D}) \leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$$

□

**Theorem 1.5.** *Let $(f_n : [a, b] \to \mathbb{R})$ be a sequence of differentiable functions such that $\exists x_0 \in [a, b]$ s.t. $f_n(x_0)$ converges. Suppose $f_n'$ converges uniformly on $[a, b]$, then $f_n$ converges uniformly to a differentiable function $f$, and $f' = \lim_n f_n'$.*

*Proof.* (c.f. 2011 and 2010 analysis II) Define auxilliary function

$$g_{c,n}(x) = \begin{cases} \dfrac{f_n(x) - f(c)}{x - c} & x \neq c \\ f_n'(c) & x = c \end{cases}$$

Prove that $(g_{c,n})$ is uniformly Cauchy (using MVT) and converges to $g_c(x) = (f(x) - f(c))/(x - c)$. Interchange limit. □

Alteranative (weaker) version: Suppose $f_n' \to g$ uniformly. If $f_n$ are continuously differentiable, then $f_n$ converges ptwise to a diff functions.

Have $f_n(x) = f_n(x_0) + \int_{x_0}^x f_n'$ by FTC. Unif. conv. implies that $\int_{x_0}^x f_n' \to \int_{x_0}^x g$, so have $f_n$ converges pointwise, and the pointwise limit is diff by FTC.

**Theorem 1.6.** *Continuous functions on closed bounded intervals are uniformly continuous.*

*Proof.* 1) By compactness (need Heine-Borel); OR 2) by contradiction (Use Bolzano-Weierstrass).

Special case of "Continuous functions on a sequentially compact metric space are uniformly continuous". □

**Theorem 1.7** (Riemann integrability of continuous functions)**.** *Continuous functions on closed bounded intervals are Riemann integrable*

*Proof.* Given $\epsilon > 0$, can find $\delta > 0$ s.t. $|f(x) - f(y)| < \epsilon/(b-a)$ whenever $|x - y| < \delta$. Take a dissection $\mathcal{D}$ of $[a, b]$ s.t. each subinterval has length $< \delta$, then $U(f, \mathcal{D}) - L(f, \mathcal{D}) < \epsilon$. □

**Theorem 1.8** (Weierstrass M-test)**.** *If there exists a sequence $(M_n)$ of non-negative real numbers s.t. $|f_n| \leq M_n$ for all $n \in \mathbb{N}$ and $\sum_n M_n$ converges, then $\sum_n f_n$ converges uniformly.*

*Proof.* Prove that the sequence of partial sums is uniformly Cauchy. □

**Proposition 1.9** (Properties of power series)**.** *Let $\sum a_n x^n$ be a complex power seires with radius of convergence $R$. Define $f : x \mapsto \sum a_n x^n$. Then*

1. *$f$ is cts*
2. *$f$ is integrable term-by-term*
3. *$f$ is differentiable and for all $x \in D(0, R)$, the derivative $f'(x)$ is given by term-by-term differentiation.*

*Proof.* Weierstrass M-test for local uniform convergence, then the rest follows. □

**Theorem 1.10** (Dini's theorem)**.** *Let $X$ be a compact metric space and $(f_n)$ a sequence of continuous real valued functions on $X$ which is decreasing. If $f_n \to f$ pointwise to a cts function $f$, then the convergence is uniform.*

**Theorem 1.11** (Arzela-Ascoli (non-examinable but useful))**.** *s*

# 2 Metric space

**Definition 2.1.** Completeness: every cauchy sequence converges within the space.
Sequential compactness: every sequence has a convergent subsequence that converges in the set.
Total boundedness: For all $\epsilon > 0$, there exists finite $A \subseteq X$ s.t. $\forall x \in X$, $\exists a \in A$, $d(x, a) < \epsilon$. (For any $\epsilon > 0$, $X$ can be covered by a finite collection of open $\epsilon$-balls centered in $X$.)

**Proposition 2.2.** *(Metric space) $\epsilon - \delta$ continuity $\Leftrightarrow$ "open sets"/Nbd continuity.*

*Proof.* $f : X \to Y$.

First show that continuity at $a$ is equivalent to $f^{-1}(N)$ being a nbd of $a$ for all nbd $N$ of $f(a)$.

Use this to show that if $G$ is open in $Y$ then $G$ is a nbd of every point $y \in G$. Use nbd continuity. Conversely, definition chasing. □

**Theorem 2.3** (Contraction mapping theorem)**.** *Let $(X, d)$ be a non-empty, complete, metric space. If $f : X \to X$ is a contraction (i.e., there exists $K < 1$ s.t. $d(f(x_1), f(x_2)) \leq K d(x_1, x_2)$ for all $x_1, x_2 \in X$), then $f$ has a unique fixed point in $X$.*

*Proof.* $X \neq \varnothing$, so take $x_0 \in X$. Construct a sequence recursively by $x_{n+1} = f(x_n)$, then if $\Delta = d(x_0, x_1)$, we must have $d(x_n, x_{n+1}) \leq K^n \Delta$. So, for $N \leq m < n$

$$d(x_n, x_m) \leq \sum_{i=m}^{n} d(x_i, x_{i+1}) \leq \Delta \sum_{i=m}^{n} K^i \leq \frac{K^N \Delta}{1-K} \to 0$$

as $N \to \infty$, so $(x_n)_{n \geq 0}$ is a Cauchy sequence, which converges in $X$ by completeness. Let $x = \lim_n x_n$, then $f(x) = x$ by uniqueness of limit. If there is another fixed point $y$, then $d(x, y) = d(f(x), f(y)) \leq K d(f(x), f(y))$, so the only way this holds is $d(x, y) = 0$ so $x = y$. □

**Proposition 2.4.** *$l^p$ is complete. (c.f. Kolmogorov, Introductory Real Analysis)*

*Proof.* $l^p$ is the space of sequences $(x_n)$ s.t $\sum |x_n|^p < \infty$. Metric is given by $d(x,y) = (\sum |x_n - y_n|^p)^{1/p}$. Given a Cauchy sequence $(x^{(n)})_{n\in\mathbb{N}}$, we see that each coordinates must be a Cauchy sequence in $\mathbb{R}$, so we have a pointwise limit $x$. For all $M$, we have the partial sum $\sum_{k=1}^{M} |x_k^{(n)} - x_k^{(m)}|^p < \epsilon$ for sufficiently large $m, n$, Let $m \to \infty$, and then let $M \to \infty$, we get $\sum_{k=1}^{\infty} |x_k^{(n)} - x_k|^p \leq \epsilon$, then use Minkowski's inequality ($p = 2$ reduces to Cauchy-Schwarz) to conclude that the convergence of $\sum |x_k^{(n)}|^p$ and $\sum |x_k^{(n)} - x_k|$ implies the convergence of $\sum |x_k|^p$. So the pointwise limit is in $l^p$. Then we can invoke metric notation and deduce completeness. □

*Lemma* 2.5. *$U \subseteq \mathbb{R}^n$ closed and bounded, $V \subseteq \mathbb{R}^m$ closed, then $C(U,V)$ (equipped with the uniform metric) is complete*

*Proof.* Take a Cauchy sequence, then this sequence of (cts) functions is uniformly Cauchy, hence uniformly converges to a cts function. Note that $V$ is closed, so the image of the limit function lies in $V$. □

**Theorem 2.6** (Picard-Lindelöf)**.** *Let $a, b \in \mathbb{R}$ with $a < b$. Let $t_0 \in (a,b)$. Let $y_0 \in \mathbb{R}^n$. Let $\phi : [a,b] \times \overline{B_\delta(y_0)} \to \mathbb{R}^n$ be continuous, and s.t. $\exists K > 0$ s.t. $\forall t \in [0,1]$ and $\forall x, y \in \overline{B_\delta(y_0)}$, $\|\phi(t,x) - \phi(t,y)\| \leq K\|x - y\|$. Then, there exists $\epsilon > 0$ s.t. the IVP $f'(t) = \phi(t, f(t)), f(t_0) = y_0$ has a unique solution on $[t_0 - \epsilon, t_0 + \epsilon] \subseteq [a,b]$.*

*Proof.* Consider the operator $T : C([t_0 - \epsilon, t_0 + \epsilon], \overline{B_\delta(y_0)}) \to ?$ (uniform metric assumed) defined by

$$T(f) = y_0 + \int_{t_0}^{x} \phi(t, f(t))dt$$

We want $T$ to be a self map, i.e., $\mathrm{im}(Tf) \subseteq \overline{B_\delta(y_0)}$, so consider

$$\|T(f) - y_0\| \leq \int_{t_0}^{x} \|\phi(t, f(t))\|dt \leq M|x - t_0| \leq M\epsilon \leq \delta$$

So we choose $\epsilon \leq \delta/M$, where $M$ is the bound of $\phi$ on this compact interval. Now, consider

$$\|T(f), T(g)\| \leq \int_{t_0}^{x} \|\phi(t, f(t)) - \phi(t, g(t))\|dt \leq \int_{t_0}^{x} K\|f(t) - g(t)\|dt \leq K|x - t_0|d(f,g) \leq K\epsilon d(f,g)$$

Choose $\epsilon$ small enough so that $K\epsilon < 1$, then we have a contraction mapping. Use Lemma 2.4 and CMT to deduce the existence of a unique fixed pt. By FTC, the unique fixed point is precisely the solution to the IVP. □

**Theorem 2.7** (Improved version of Picard-Lindelöf (Sheet 2 Q9))**.**

**Theorem 2.8.** *Let $(X,d)$ be a metric space. TFAE,*

- *$(X,d)$ is compact;*
- *$(X,d)$ is sequentially compact;*
- *$(X,d)$ is complete and totally bounded.*

*If $X \subseteq \mathbb{R}^n$ (equipped with Euclidean metric), then*

- *$(X,d)$ is closed and bounded.*

*Proof.* Sequential compactness ⇔ complete and totally bounded:
⇒: Suppose not complete but sequentially compact, then have a cauchy sequence which doesn't converge in $X$, but it has a subsequence which converges in $X$, so the sequence converges in $X$. Contradiction. Suppose not totally bounded, then

$$\exists \epsilon \forall A \subseteq X \text{ finite }, \exists x \in X \forall a \in A, d(x,a) \geq \epsilon$$

Pick $x_1 \in X$, and recursively let $A_n = \{x_1, \ldots, x_n\}$, then can pick $x_{n+1}$ s.t. $d(x_i, x_{n+1}) \geq \epsilon$ for all $i \leq n$. This gives a sequence s.t. for all $i, j \in \mathbb{N}$ $i \neq j \Rightarrow d(x_i, x_j) \geq \epsilon$, so it has no convergent subsequence.
⇐: (Similar to a bisection proof) Suppose complete and totally bounded. Let $(x_n)$ be a sequence in $X$. Can find $A_1$ finite such that all $X \subseteq \bigcup_{a\in A_1} B_{1/2}(a)$. So there exists $a_1 \in A_1$ s.t. $d(x_n, a_1) < 1/2$

infinitely often, so pick a subsequence $(x_{n_{1,j}})_{j\in\mathbb{N}}$. Keep going and for each $k \in \mathbb{N}$, can find subsequence $(x_{n_{2,j}})$ of $(x_{n_{1,j}})$ such that they all lie in $B_{1/4}(a_2)$. Repeat this process. Obtain a bunch of sequences. Use diagonalization to get a Cauchy sequence, which converges in $X$ by completeness, so $X$ is sequentially compact.

Compactness $\Leftrightarrow$ sequential compactness: $\Rightarrow$: If compact but not sequentially compact, then find a sequence $(x_n)$ with no convergent subsequence. Then, for all $a \in X$, there exists $G_a$ open which contains only finitely many $x_n$ (Otherwise we would have a convergent subsequence). $\bigcup_{a\in X} G_a$ gives an open cover, which reduces to a finite subcover, but then $X$ can only contain finitely many $x_n$, contradiction. $\Leftarrow$: Want to use total boundedness to find a finite subcover. Let $C$ be an open cover of $X$. Claim:

$$\exists \delta > 0, \forall a \in X, \exists G \in C : B_\delta(a) \subseteq G$$

If not then for all $n \in \mathbb{N}$, there exists $x_n \in X$ witht the property that $\forall G \in C$, $B_{1/n}(x_n) \nsubseteq G$. Sequential compactness gives a convergent subsequence. Let $x$ be the limit, then $x \in G_0 \in C$, meaning that $B_\epsilon(x) \subseteq G_0$, but then can go sufficiently far down the subsequence so that everyone lies in $B_\epsilon(x)$ and $1/n < \epsilon$. Contradiction. So such $\delta$ exists. By total boundedness (implied by sequential compactness) can cover $X$ by a finite collection of $\delta$-balls, so can take finitely many $G \in C$ to cover $X$, which is a finite subcover of $C$.

For subsets of Euclidean spaces. If closed and bounded, then Bolzano-Weierstrass implies that $X$ is sequentially compact. Conversely, if not closed then find a convergent sequence that doesn't converge in $X$, then it can't have subsequences converging in $X$ by uniqueness of limit. If not bounded, then obviously have unbounded sequence which has no convergent subsequence. □

**Corollary 2.9** (Heine-Borel)**.** *A subsets of Euclidean space is compact iff it's closed and bounded. [This implies that* $[0, 1]$ *(usual metric) is compact.]*

*Proof.* □

**Proposition 2.10.** *Compact metric spaces are complete.*

*Proof.* Compact implies sequentially compact, so any cauchy sequence has a convergent subsequence, then the whole sequence must converge. □

**Theorem 2.11.** *Continuous functions on a compact metric space is uniformly continuous*

*Proof.* For each point in $x \in K$, $\exists \delta_x > 0$ s.t. $\epsilon$-$\delta$ condition holds. Take union of $\delta_x$-balls (open cover). Pass to a finite subcover. □

**Theorem 2.12.** *Let* $(X, d)$ *be a sequentially compact metric space, and* $f : X \to X$ *isometry, then* $f$ *is bijective.*

*Proof.* Injectivity is trivial. Assume $y \notin f(X)$, then by sequential compactness and continuity, $\exists \epsilon > 0$ s.t. $d(f(x), y) \geq \epsilon$. Let $x_0 = x$ and define $x_{n+1} = f(x_n)$, we have for $i < j$, $d(x_i, x_j) = d(x_0, x_{j-i}) \geq \epsilon$, so $(x_i)$ has no convergent subsequences, contradicting sequential comapctness. □

# 3 Topology

**Proposition 3.1.** *Open set continuity (Continuity)* $\implies$ *sequential continuity. (The converse is false even for Hausdorff space)*

**Proposition 3.2.** $[0, 1]$ *is connected.*

*Proof.* (IVT banned) Suppose not, then have $U, V \subseteq [0, 1]$ disjoint open subsets s.t. $U \cup V = [0, 1]$. WLOG, $1 \in V$. Let $\eta = \sup U$. Have $\eta \neq 0, 1$ otherwise $U = \{0\}$ is not open or 1 is not an interior point of $V$.

Suppose $0 < \eta < 1$, then if $\eta \in U$, then get contradiction as there is an open nbd of $\eta$ in $U$. If $\eta \in V$, then the same argument (go downward instead of upward) gives a contradiction. So $[0, 1]$ is connected. □

*Proof.* (IVT not banned). Map an open set to 0 and the other to 1, get a cts function, IVT implies that it must hit 1/2 somewhere, contradiction. □

**Proposition 3.3.** *Path-connectedness implies connectedness. For subsets of $\mathbb{R}^n$ with the Euclidean topology, connectedness implies path-connectedness.*

**Theorem 3.4.** *Image of connected (resp. compact) sets under continuous functions are connected (resp. compact).*

**Theorem 3.5.** *Continuous real-valued function on sequentially compact sets is bounded and attains its bound.*

**Corollary 3.6.** *$(X, d)$ metric space and $K \subseteq X$ sequentially compact, then for all $x \in X \setminus K$, there exists $\epsilon > 0$ s.t. for all $y \in K$, $d(x, y) \geq \epsilon$ and $\exists z \in K$ s.t. $d(x, z) = \epsilon$*

*Proof.* Consider $y \mapsto d(x, y)$. □

**Proposition 3.7.** *Finite Cartesian product of connected (resp. compact) sets are connected (resp. compact).*

*Proof.* □

**Theorem 3.8.** *Intersection of nested sequence of closed comapct sets is non-empty.*

*Proof.* (c.f. Sheet 3) □

*Lemma* 3.9. Closed subset of compact space is compact; compact subset of Hausdorff space is closed.

*Proof.* $K$ compact. $X \subseteq K$ closed. $X \setminus K$ open Take an open cover of $X$, then by adjoining $X \setminus K$ if necessary, we get an open cover of $K$, which has a finite subcover, then remove $X \setminus K$ if necessary, we get a finite subcover of $X$.

$Y$ Hausdorff, $X \subseteq Y$ compact. Pick $y \in Y \setminus X$, then for each $x \in X$ there exists nbd $U_x, V_y$ s.t. $U_x \cap V_y = 0$, $\{U_x\}$ is an open cover of $X$, so pass to a finite subcover, then construct $V \subseteq Y \setminus X$ by taking intersection, so $Y \setminus X$ is open. □

**Theorem 3.10** (Topological inverse function theorem)**.** *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

*Proof.* It suffices to prove that this is a closed map. Given a closed set in the domain, then it's compact, its image is a compact subset of Hausdorff space, hence closed. □

**Theorem 3.11** (Universal property of quotient topology)**.** *Let $(X, \tau)$ be a top. space and $\sim$ an equiv. relation on $X$. Let $q : X \to X/\sim$ be the quotient map and let $f : X \to Y$ be a cts function respecting $\sim$, then $\exists!$ $\tilde{f}$ cts s.t. $f = \tilde{f} \circ q$. [$X/\sim$ is equipped with the quotient topology.]*

*Proof.* First prove that there is a well-defined $\tilde{f}$, and it is clearly unique by commutativity requirement. Then use definition of quotient topology to prove continuity. Pullback an open set $G \subseteq Y$ via $(q \circ \tilde{f})$ get an open set in $X$ by continuity of $f$. Quotient topology says $\tilde{f}^{-1}(G)$ is open in the quotient iff $q^{-1}\tilde{f}^{-1}(G)$ is open in $X$ which is true by continuity of $f$. □

### 3.1 Equivalence of topologies

## 4 Multivariate differentiation

**Proposition 4.1** (Operator norm)**.** *$\alpha \in L(\mathbb{R}^n, \mathbb{R}^m)$, define $\|\alpha\| = \sup\{\|\alpha(x)\| : \|x\| = 1\}$.*

- $\|\alpha\| \geq 0$ *with equality iff* $\alpha = 0$
- $\|\lambda\alpha\| = |\lambda|\|\alpha\|$ *for scalars* $\lambda$
- $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$.
- $\|\alpha(x)\| \leq \|\alpha\|\|x\|$
- *If* $\gamma \in L(\mathbb{R}^m, \mathbb{R}^k)$, *then* $\|\gamma \circ \alpha\| \leq \|\alpha\|\|\beta\|$

**Proposition 4.2.** *"Equivalence" of Operator norm and Euclidean norm*

*Proof.* Bound by some obvious inequalities. □

**Proposition 4.3** (Chain rule)**.** *Let $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$. If $f : U \to \mathbb{R}^n$, $g : V \to \mathbb{R}^k$, and $f(U) \subseteq V$ and $f$ is diff at $a \in U$, $g$ is diff at $f(a)$, then $g \circ f$ is differentiable at $a$ and $D(g \circ f)|_a = Dg|_{f(a)} \circ Df|_a$.*

*Proof.* Algebra bash. Write $b = f(a)$ and

$$\begin{cases} f(a+h) = f(a) + S(h) + \epsilon(h)\|h\| \\ g(b+k) = g(b) + T(k) + \delta(k)\|k\| \end{cases}$$

So

$$g(f(a+h)) = g(b + S(h) + \epsilon(h)\|h\|)$$

Expand. will get two terms. One of them is $o(\|h\|)$ by direct computation (divide by norm $h$, goes to zero) The other term goes to zero by estimating using operator norm. □

**Theorem 4.4.** *If all partial derivatives exist and are continuous in a nbd of a point, then $f$ is differentiable.*

*Proof.* WLOG, $m = 1$ (looking at each coordiante separately). For general $n$ apply MVT multiple times. □

*Remark* 1. In fact, we only require $n - 1$ partial derivatives to be continuous in a nbd of that point.

**Theorem 4.5** (Mean value inequality)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and let $a, b \in \mathbb{R}^n$. Suppose $f$ is diferentiable at $z$ for all $z \in [a, b] = \{a + t(b - a) : t \in [0, 1]\}$. Then,*

$$\|f(b) - f(a)\| \leq \|b - a\| \sup_{z \in [a,b]} \|Df|_z\|$$

*Proof.* Define $F : [0, 1] \to \mathbb{R}$ by

$$F(t) = f(a + t(b - a)) \cdot (f(b) - f(a))$$

Notice that $F(1) - F(0) = \|f(b) - f(a)\|^2$. $F$ is differentiable with

$$DF|_t(h) = (f(b) - f(a)) \cdot Df|_{a+t(b-a)}(h(b - a))$$

(chain rule), so $F'(t) = (f(b)-f(a)) \cdot Df|_{a+t(b-a)}(b-a)$. We apply MVT to conclude that $|F(1)-F(0)| = |F'(\xi)|$ for some $\xi \in (0, 1)$, so

$$\begin{aligned} \|f(b) - f(a)\|^2 &= |F(1) - F(0)| \\ &= |F'(\xi)| \\ &\leq \|f(b) - f(a)\|\|Df|_{a+\xi(b-a)}(b - a)\| \\ &\leq \|f(b) - f(a)\|\|Df|_{a+\xi(b-a)}\|\|b - a\| \\ &\leq \|f(b) - f(a)\|\|b - a\| \sup_{z \in [a,b]} \|Df|_z\| \end{aligned}$$

where we have used Cauchy-Schwarz and properties of operator norm. □

**Corollary 4.6.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and let $G \subseteq \mathbb{R}^n$ be open and connected. Suppose for all $z \in G$, $f$ is differentiable at $z$ with $Df|_z$ being the zero map. Then $f$ is constant on $G$.*

**Theorem 4.7** (Symmetry of mixed partial derivatives)**.** *Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ and $z \in \mathbb{R}^n$. If $D_i D_j f$ and $D_j D_i f$ exists and cts in a nbd of $z$ and are cts at $z$, then $D_i D_j f = D_j D_i f$.*

*Proof.* WLOG, $n = 2$, $m = 1$. Consider $\Delta_h = f(x+h, y+h) - f(x+h, y) - (f(x, y+h) - f(x, y))$. Use differentiability on each bracket and apply MVT. By similar argument can deduce two formula of $\Delta_h$ in terms of mixed partial. Let $h \to 0$. □

**Theorem 4.8** (Second order Taylor's theorem)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be twice differentiable, then*

$$f(a + h) = f(a) + Df|_a(h) + \frac{1}{2} D^2 f|_a(h, h) + o(\|h\|)^2$$

*Proof.* WLOG, $m = 1$. Define

$$\varphi_h(t) = f(a + th) - f(a) - tDf|_a(h) + \frac{t^2}{2} D^2 f|_a(h, h)$$

Note that $\varphi_h(1) - \varphi_h(0)$ is the expression we want. Want to prove that this difference is $o(\|h\|^2)$, so apply MVT and use the definition of second derivative to manipulate. $\square$

**Theorem 4.9** (Inverse function theorem). *Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $a \in \mathbb{R}^n$, and $b = f(a)$. Suppose there exists ome open nbd $A$ of a s.t. $f$ is continuouslu differentiable at every $x \in A$ and $Df|_a$ is non-singular. Then there are open nbd $U$ of $a$ and $V$ of $b$ s.t. $f|_U : U \to V$ is a homeomorphism. Moreover, if $g : V \to U$ is the inverse, then $g$ is continuously differentiable at all $y \in V$ with $Dg|_y = (Df|_x)^{-1}$, where $y = f(x)$.*

*Proof.* Step 0: Let $A = Df|_a$. For each $y \in \mathbb{R}^n$ define $\varphi_y(x) = x + A^{-1}(y - f(x))$. Can find $\lambda > 0$ s.t. $2\lambda\|A^{-1}\| = 1$. By continuity of derivative, can find an open ball $U$ about $a$ s.t. $\|A - Df|_x\| < \lambda$ for all $x \in U$.

Step 1: For each $y_0 = f(x_0) \in f(U)$, there exists $\delta > 0$ and $\eta > 0$ s.t. for all $y \in B_\eta(y_0)$, $\varphi_y|_{\overline{B_\delta(x_0)}}$ is a contraction mapping. First, note that on $U$, $D\varphi_y|_x = I - A^{-1} \circ Df|_x = A^{-1}(A - Df|_x)$. The norm of this thing is $\leq \lambda(1/2\lambda) = 1/2$ on $U$, so by mean value inequality, we have (for all $x_1, x_2 \in U$)

$$\|\varphi_y(x_1) - \varphi_y(x_2)\| \leq \frac{1}{2}\|x_2 - x_1\|$$

This implies that $\varphi_y$ can have at most one fixed point in $U$, i.e., $f(x) = y$ has at most one solution for $x$ in $U$. Second, pick $\delta > 0$ s.t. $\overline{B_\delta(x_0)} \subseteq U$, and let $\eta = \lambda\delta$. Now, for $x \in \overline{B_\delta(x_0)}$ and $y \in B_{\lambda\delta}(y_0)$,

$$\begin{aligned}\|\varphi_y(x) - x_0\| &\leq \|\varphi_y(x) - \varphi_y(x_0)\| + \|\varphi_y(x_0) - x_0\| \\ &\leq \frac{1}{2}\|x - x_0\| + \|A^{-1}\|\|y - y_0\| \leq \frac{\delta}{2} + \frac{\lambda\delta}{2\lambda} = \delta\end{aligned}$$

Contraction mapping theorem then implies that there exists a unique $x \in \overline{B_\delta(x_0)}$ s.t. $f(x) = y$. Also, we see that if $y_0 \in f(U)$, then $B_{\lambda\delta}(y_0) \subseteq f(U)$, i.e., $f(U)$ is open. This shows that $f|_U : U \to f(U)$ is a continuous bijection from open set $U$ to open set $f(U)$.

Step 3: Let $g = (f|_U)^{-1}$. We wish to show that $g$ is continuous. Let $y_1 = f(x_1)$, $y_2 = f(x_2)$ for some $x_1, x_2 \in U$. Note that for $y \in \mathbb{R}^n$

$$\frac{1}{2}\|x_1 - x_2\| \geq \|\varphi_y(x_1) - \varphi(x_2)\| = \|x_1 - x_2 + A^{-1}(y_2 - y_1)\| \geq \|x_1 - x_2\| - \|A^{-1}(y_2 - y_1)\|$$

Rearranging, get

$$\|g(y_1) - g(y_2)\| = \|x_1 - x_2\| \leq 2\|A^{-1}(y_2 - y_1)\| \leq 2\|A^{-1}\|\|y_1 - y_2\|$$

So $g$ is continuous $\square$

# 5 Useful estimates/inequalities

- $e^x \geq x^n/n!$ for $x > 0$

# 6 Counterexamples

**Example 1** (Pointwise limit of cts func being discontinuous). $f_n(x) = x^n$ *on* $[0, 1]$ *OR* $f_n(x) = (1 - x^2)^n$ *on* $[-1, 1]$

**Example 2** (Limit of integral != integral of ptwise limit). *Increasing spike.*

**Example 3** (Uniform limit of differentiable functions being non-differentiable). $f_n(x) = \sqrt{x^2 + 1/n}$ *on* $[-1, 1]$

**Example 4.** *Uniform limit of differentiable functions being differentiable but the limit is incorrect* $f_n(x) = \dfrac{x}{1 + nx^2}$, *then* $\lim f_n'(x) = f'(x)$ *is true iff* $x \neq 0$.

**Example 5** (bounded but not totally bounded). *Closed unit ball in* $l^2$.

**Example 6** (Function on a Hausdorff space which preserve limits but is discontinuous).

**Example 7** (Connected but not path-connected). $X = \{(x, \sin(1/x)) : x \in (0, 1]\} \cup \{(0, y) : y \in [0, 1]\}$.