CA

1 Basic stuff about holomorphic functions

Theorem 1.1 (Cauchy-Riemann equation). *Let $f : U \to \mathbb{C}$ be a function on an open set $U \subseteq \mathbb{C}$. Then $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic at $z = c + id \in U$ with derivative $p + iq$ if and only if u, v are real differentiable at (c,d) and they satisfy the Cauchy-Riemann equation, i.e.,*

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

Remark 1. u, v are said to be harmonic conjugate to each other.

Proposition 1.2 (Conformality). *If $f : U \to \mathbb{C}$ is holo'c and $f'(w) \neq 0$, then f preserves angles at $z = a$.*

Proof. Take two paths γ_1, γ_2 s.t. $\gamma_1(0) = \gamma_2(0) = w$. Have $\theta = \text{Arg}(\gamma_2'(0)) - \text{Arg}(\gamma_1'(0))$. f is angle preserving because $\text{Arg}(f \circ \gamma_j)'(0)) = \text{Arg}(\gamma_j'(0) f'(w)) = \text{Arg}(\gamma_j'(0)) + \text{Arg}(f'(w)) + 2n\pi$. (valid since $f'(w) \neq 0$ by assumption, so we have well-defined argument) □

2 Complex integrals

Theorem 2.1 (FTC). *If $f : U \to \mathbb{C}$ and $U \subseteq \mathbb{C}$ open, and there exists F on U s.t. $F' = f$, then for any curve $\gamma : [0,1] \to U$,*

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a))$$

Proof. Direct computation. In the final step split into real and imag part and use real analysis. □

Theorem 2.2 (Antiderivative thm (partial converse to FTC)). *If $f : D \to \mathbb{C}$ is continuous on a domain D, and $\int_\gamma f = 0$ for all closed curves, then f has a primitive on D.*

Theorem 2.3 (Goursat's theorem). *$f : U \to \mathbb{C}$ holo'c, $U \subseteq \mathbb{C}$ open, then $\int_{\partial T} f = 0$ for all triangles $T \subseteq U$.*

Proof. Pick some $T \subseteq U$ (Note that T is closed and bounded hence compact). Let $I = \int_{\partial T} f$ and $L = \text{length}(\partial T)$ Subdivide into $T_1, ..., T_4$ using midpoints of edges, then the integral along the boundary of one of them is $\geq 1/4I$, call it $T^{(1)}$. Subdivide $T^{(1)}$ and construct $T^{(2)} \geq 1/4 \int_{\partial T^{(1)}} f \geq 1/16I$. Repeat this process, we get a sequence

$$T^{(1)} \supseteq T^{(2)} \supseteq \dots$$

We have $\text{length}(T^{(j)}) \leq 2^{-j}L$, so $\text{diam}(T^{(j)}) \to 0$ as $j \to \infty$. Claim $\bigcap_i T^{(i)} \neq \varnothing$ (intersection of nested sequence of compact sets). Choose w in this big intersection.

Let $\epsilon > 0$ be given. f is holo'c at w, so can pick $\delta > 0$ s.t. $|f(z) - f(w) - (z-w)f'(w)| < \epsilon|z-w|$ whenever $|z - w| < \delta$. Also, can pick N s.t. $T^{(n)} \subseteq D(w, \delta)$ for all $n \geq N$ (possible because diam goes to zero).

$$4^{-n}I \leq \left|\int_{\partial T^{(n)}} f\right| = \left|\int_{\partial T^{(n)}} (f(z) - f(w) - (z-w)f'(w))dz\right| \leq 2^{-n}L\epsilon \sup_{\partial T^{(n)}} |z - w| \leq 4^{-n}L^2\epsilon$$

Rearrange. □

Proposition 2.4. *Let $S \subseteq U$ be a finite subset of a domain and $f : U \to \mathbb{C}$ holomorphic away from S and f is continuous on U, then for any triangle $T \subseteq U$, $\int_{\partial T} f = 0$.*

Proof. WLOG assume $|S| = \{a\}$. Pick some $T \subseteq U$ which contains a. By subdivision, can find smaller T' s.t. $a \in T' \subseteq T$, then Goursat implies $\int_T f = \int_{T'} f$. Estimate $|\int_{T'} f| \le \text{length}(T') \sup_{z \in \partial T'} |f(z)|$. The sup term is bounded by continuity, so RHS goes to 0 as T' shrinks. □

Theorem 2.5 (Cauchy's theorem for convex/star-convex domain). *If $f : U \to \mathbb{C}$ is cts, and holo'c away from finitely points, then $\int_\gamma f = 0$ for any closed curves γ.*

Proof. Antiderivative theorem for star-convex domain is true with weaker hypothesis: $\int_{\partial T} f = 0$ for all triangles $T \subseteq U$. (Proof exactly the same)

Preceding theorem shows that $\int_{\partial T} f = 0$ for any triangle T. By antiderivative theorem for star-convex comain, f has an antiderivative on U. Apply FTC. □

Theorem 2.6 (Cauchy integral formula on a disk (basic)). *Let $U \subseteq \mathbb{C}$ be a domain. If $f : U \to \mathbb{C}$ is holo'c and $\overline{D(a,r)} \subseteq U$, then for all $z \in D(a,r)$,*

$$f(z) = \frac{1}{2\pi i} \int_{\partial D(a,r)} \frac{f(w)}{w - z} dw$$

Proof. By Cauchy's theorem on a disk

$$\int_{\partial D(a,r)} \frac{f(w) - f(z)}{w - z} dw = 0$$

Have $|w - a| = r > |z - a|$, so geometric expansion works.

$$\frac{1}{w - z} = \frac{1}{w - a - (z - a)} = \frac{1}{(w - a)(1 - (z - a)/(w - a))} = \sum_{n \ge 0} \frac{(z - a)^n}{(w - a)^{n+1}}$$

Swap limit using uniform convergence. □

Corollary 2.7 (Mean value property).

$$f(a) = \int_0^1 f(a + re^{2i\pi t}) dt$$

The proof is essentially by CIF.

Theorem 2.8 (Liouville's theorem). *Bounded entire functions are constant.*

Proof. Pick some $z \neq 0$ and $r > |z|$.

$$\begin{aligned} |f(z) - f(0)| &= \frac{1}{2\pi} \left| \int_{D(0,r)} f(w) \left(\frac{1}{w - z} - \frac{1}{w} \right) dw \right| \\ &= \frac{|z|}{2\pi} \left| \int_{D(0,r)} \frac{f(w)}{w(w - z)} \right| \\ &\le \frac{|z|}{2\pi} 2\pi r \sup_{|w| = r} \frac{|f(w)|}{r|w - z|} \\ &\le \sup_{|w| = r} \frac{|z| M}{|w - z|} \to 0 \end{aligned}$$

as $r \to \infty$. □

Theorem 2.9 (Generalization of Liouville (not covered)). *Entire functions with sublinear growth are constant.*

The proof is entirely the same. Just replace M with bound of the form $M(1 + |w|^\alpha)$, where $\alpha \in (0, 1)$.

Theorem 2.10 (Fundamental theorem of algebra). *If $p(x) \in \mathbb{C}[x]$ is non-constant, then $p(x)$ has a root in $\mathbb{C}$.*

Proof. If p has no root, then consider $\frac{1}{f(z)}$. It's entire and bounded (Use limit to bound everything except on a closed disk, then use compactness). Contradicting Liouville. □

Theorem 2.11 (local maximum (modulus) principle). *Let $f : D(a,r) \to \mathbb{C}$ be holo'c. If $|f(z)| \le |f(a)|$ for all $z \in D(a,r)$, then f is constant on $D(a,r)$.*

Proof. Mean value property,

$$|f(a)| = \left| \int_0^1 f(a + \rho e^{2\pi i t}) dt \right| \le \sup_{t \in [0,1]} |f(a + \rho e^{2\pi i t})| \le f(a)$$

for all $0 < \rho < r$. Inequality must be equality, so $|f|$ must be constantly equal to $|f(a)|$, so f is constant (C-R equation or Liouville). □

3 Expansions

Theorem 3.1 (Taylor series representation). *$f : D(a,r) \to \mathbb{C}$ holo'c Then f is represented by a convergent power series on $D(a,r)$*

$$f(z) = \sum_{n \ge 0} c_n (z-a)^n, \; c_n = f^{(n)}(a)/n! = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(w)}{(w-a)^{n+1}} dw$$

for any $|z| < \rho < r$.

Proof. Let $|z - a| < \rho < r$. Apply CIF for disks and geometric expansion (swap limit by unif convergence). □

So holo'c functions are analytic.

Proposition 3.2 (CIF for derivatives). *Let f be holo'c on U and $\overline{D(a,r)} \subseteq U$. Then for all $z \in D(a,r)$,*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D(a,r)} \frac{f(w)}{(w-z)^{n+1}} dw$$

Proof. By induction. Consider $f(w)/(w-z)^{n+1}$ and differentiate with respect to w. Use antiderivative thm + induction hypothesis. □

Theorem 3.3 (Morera's theorem). *$f : U \to \mathbb{C}$. If $\int_\gamma f = 0$ for all closed curves γ, then f is holo'c on U.*

Proof. Antiderivative thm + analyticity. □

Theorem 3.4 (Laurent series representation). *If f is holo'c on an annulus $A = \{z \in \mathbb{C} : r < |z-a| < R\}$, where $0 \le r < R \le \infty$, then*

- *f has a unique convergent expansion (Laurent series) on A, namely*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

- *for any $r < \rho' \le \rho < R$, the Laurent series converges uniformly $\{\rho' \le |z-a| \le \rho\}$.*
- *For any $r < \rho < R$, coefficients are given by*

$$c_n = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(w)}{(w-a)^{n+1}} dw$$

Theorem 3.5 (Residue theorem). *Let f be meromorphic in a domain D and γ is a closed curve which is homologous to 0 in D. Assume no poles of f lie on γ and only finitely many poles at $\{a_1, ..., a_m\}$ of f has $I(\gamma, a_i) \ne 0$, then*

$$\int_\gamma f = 2\pi i \sum_{i=1}^m I(\gamma; a_i) \operatorname{Res}_{z=a_i} f(z)$$

Proof. Subtract all principal parts from f, then the resulting function is holomorphic (has removable singularities only) in the domain obtained by removing singularities with $I(\gamma, a) \neq 0$ (γ is still homologous to zero in this new domain). Apply Cauchy's theorem (generalized version). $\square$

Theorem 3.6 (Jordan's lemma)**.** *Suppose f is holo'c for $|z| > r$ for some $r > 0$ and assume that $zf(z)$ is bounded (Or simpler $|f(z)| \to 0$ as $|z| \to \infty$). Then for all $\alpha > 0$, we have*

$$\int_{C'_R} f(z) e^{i\alpha z} dz \to 0 \text{ as } R \to \infty$$

C'_R is $\gamma : [0, \pi] \to \mathbb{C}, t \to Re^{it}$. Similar statement holds for $\alpha < 0$ and the semicircle on lower half-plane.

Proof. Symmetry of sin and Jordan's inequality. $\square$

4 Zeros and singularities

Theorem 4.1 (Principle of isolated zeros)**.** *Let $f : D(a, r) \to \mathbb{C}$ be holo'c. f is not constantly 0. Then there exists $0 < \rho < r$ s.t. $f(z) \neq 0$ on $D(a, \rho)^*$.*

Proof. If $f(a) \neq 0$, then we are done by continuity. If $f(a) = 0$ and is a zero of some positive order, then write $f(z) = (z - a)^m g(z)$ for some holo'c g s.t. $g(a) \neq 0$ (possible by Taylor series expansion). By continuity of g, such punctured disk exists. $\square$

Theorem 4.2 (Identity theorem)**.** *Let f, g be holo'c on the domain U. Define $S = \{z \in U : f(z) = g(z)\}$. If S has an accumulation point in U, then $f(z) = g(z)$ for all $z \in U$.*

Proof. Let $h = f - g$. h is holo'c on U and has a non-isolated zero at w iff w is an accumulation point of S. Principle of isolated zeros implies that $h \equiv 0$ on some $D(w, \epsilon)$. By Taylor series representation, $h \equiv 0$ on any $D(w, r) \subseteq U$. The set $\{z \in U : \exists r > 0, h|_{D(z,r)} \equiv 0\}$ is a non-empty open subset of U. It's complement is $\{z \in U : \forall r > 0, \exists z' \in D(z, r), f(z') \neq 0\}$ which is also open (the selection condition is equivalent to $f^{(n)}(z) \neq 0$ for some n). So by connectedness, the second set is empty, so $h \equiv 0$ on U. $\square$

Corollary 4.3 (maximum modulus/global maximum)**.** *Let U be a bounded domain. If $f : \overline{U} \to \mathbb{C}$ is continuous and f is holo'c on U, then the maximum of $|f|$ is attained in $\overline{U} \setminus U$.*

Proof. $\overline{U}$ is closed and bounded so compact. $|f|$ attains max m in $\overline{U}$. Suppose $|f(z_0)| = m$ for some $z_0 \in U$, then local max principle implies that f is constant on some open disk about z_0, then identity theorem implies that f is constant on U, so f is constant on $\overline{U}$ by continuity, so $f(z) = m$ for all $m \in \overline{U} \setminus U$. $\square$

Theorem 4.4 (Argument principle)**.** *Let γ be a closed curve bounding a domain D, and let f be meromorphic on a nbd of $\gamma \cup D$. If f has no zeros or poles on γ, then*

$$I(f \circ \gamma, 0) = \int_\gamma \frac{f'}{f} dz = \# \text{ of zeros in } D - \# \text{ of poles in } D$$

(counted with multiplicities)

Proof. First prove that if f is meromorphic with a zero (resp. a pole) of order k at $z = a$. Then, $\dfrac{f'(z)}{f(z)}$ has a pole at $z = a$ with residue k (resp. $-k$) by writing $f(z) = (z - a)^k g(z)$, where g is holo'c and $g(a) \neq 0$, then compute the residue. Then use residue theorem. $\square$

Lemma 4.5 (Properties of winding number). γ closed curve. $w \mapsto I(\gamma, w)$ is a locally constant map.

Proof. Sheet 3 Q10. First show that if γ, σ are two closed curves such that for all t, $|\gamma(t) - \sigma(t)| < |\gamma(t) - w|$, then $I(\gamma, w) = I(\sigma, w)$ by considering $(\gamma - w)/(\sigma - w)$ about 0. Then use translational symmetry to deduce that if γ doesn't meet $D(w, \epsilon)$, then $\forall z \in D(w, \epsilon)$, $I(\gamma, w) = I(\gamma, z)$. $\square$

Theorem 4.6 (Local mapping degree)**.** *Let $f : D(a, R) \to \mathbb{C}$ be holo'c and non-constant with local degree $k > 0$ at $z = a$. Then for $r > 0$ sufficiently small, there exists $\epsilon > 0$ s.t. $0 < |w - f(a)| < \epsilon \implies w = f(z)$ has k simple solutions.*

Proof. By principle of isolated zero, can find $r > 0$ s.t. $f(z) - f(a) \neq 0$ and $f'(z) \neq 0$ on $\overline{D(a,r)} \setminus \{a\}$. Then $f \circ \gamma$ doesn't contain $f(a)$, so can find $D(f(a), \epsilon)$ that doesn't intersect the image of $f \circ \gamma$. For all $w \in D(f(a), \epsilon)$, $I(f \circ \gamma, w) = I(f \circ \gamma, f(a)) = k$ [c.f. sheet 3 Q10(b)]. So w has k preimages. Since $f' \neq 0$ on the punctured disk, they are all distinct. □

Corollary 4.7 (Open mapping theorem)**.** *Non-constant holo'c functions on a domain are open maps.*

Proof. Local mapping degree theorem says that If r, ϵ are sufficiently small, # preimages of w in $D(a,r) = \deg_{z=a} f(z) > 0$ for all $w \in D(f(a), \epsilon)$. In this situation, $D(f(a), \epsilon) \subseteq f(D(a,r))$. □

Theorem 4.8 (Rouche's theorem)**.** *Let γ bound a domain D, and f, g holo'c on a nbd of nbd. If $|f| > |g|$ for all $z \in \gamma$, then f and $f+g$ has the same number of zeros on D.*

Proof. $|f| > |g|$ on γ, so f and $f+g$ are nowhere 0 on γ. Apply argument principle to $h = (f+g)/f = 1 + g/f$. We have $|h-1| = |g/f| < 1$, so $h(\gamma) \in D(1,1)$, so $I(h \circ \gamma, 0) = 0$, so the number of zeros of h equals the number of poles of h in D. This is precisely saying that the number of roots of f and $f+g$ are equal (counting multiplicities). □

Remark 2. Rouche's theorem implies open mapping theorem

Proof. By principle of isolated zeros. Can find a sufficiently small $r > 0$ s.t. $f(z) - f(a) \neq 0$ on $D(a,r)^*$. Let γ be the boundary of the disk, then $|z-a| = r$. Choose $0 < \epsilon < \min\{|f(z) - f(a)|\}$. WTS $D(f(a), \epsilon) \subseteq f(D(a,r))$. Pick $w \in D(f(a), \epsilon)$. Consider $g(z) = f(z) - w$. Then $g(z) = f(z) - f(a) + f(a) - w$. Since $|f(a) - w| < \epsilon < |f(z) - f(a)|$ on γ, Rouche's theorem implies that $g(z)$ and $f(z) - f(a)$ have the same number of roots in $D(a,r)$, which is ≥ 1. Done. □

4.1 Classification of singularities

$$\begin{cases} \text{isolated} \begin{cases} \text{removable} \\ \text{poles} \\ \text{essential} \end{cases} \\ \text{non-isolated (essential)} \quad \text{[include branch point sing (CM)]} \end{cases}$$

The following theorem from sheet 2 is occasionally useful.

Theorem 4.9 (Casorati-Weierstrass)**.** *If $f : D(a,r)^* \to \mathbb{C}$ be a holo'c function which has an essential singularity at a (so a is an isolated essential singularity), then*

$$\forall w \in \mathbb{C}, \forall \epsilon > 0, \forall \delta > 0, \exists z \in D(a,\delta)^* \ \textit{s.t.}\ f(z) \in D(w, \epsilon)$$

Proof. By contradiction (c.f. sheet 2 Q9). Suppose not, then there exists $w_0 \in \mathbb{C}, \epsilon_0 > 0, \delta_0 > 0$ s.t. $\forall z \in D(a, \delta_0)^*$, $|f(z) - w_0| \geq \epsilon_0$. Consider $g(z) = 1/(f(z) - w_0)$. g is bounded and holomorphic on $D(a, \delta_0)^*$. Consider its Laurent expansion abour a. Boundedness implies that $h(z) = \sum_{n \geq 0} c_n (z-a)^n$, so $f(z) = 1/(\sum c_n (z-a)^n) + b$. By considering limit, we see that $z = a$ is either a removable singularity or a pole. Contradiction. □

5 Local uniform convergence

Proposition 5.1. *$(f_n : U \to \mathbb{C})$ is locally unif. conv. $\Leftrightarrow$ $(f_n|_K)$ is unif. conv. on any compact subset $K \subseteq U$.*

Proof. ($\Leftarrow$): Trivial. Find $D(a,r) \subseteq U$, then $\overline{D(a, r/2)} \subseteq U$ is compact. Use unif. conv. on compact subsets.
($\Rightarrow$): $K \subseteq U$ compact. For each $a \in K$, $\exists r_a > 0$ s.t. f_n conv. unif. on $D(a, r_a)$, then $\bigcup_{a \in K} D(a, r_a) \supseteq K$ so admits a finite subcover $K \subseteq \bigcup_{i=1}^n D(a_i, r_{a_i})$. Let $\epsilon > 0$ be given. For each i, there exists $N_i \in \mathbb{N}$ s.t. $n \geq N_i \implies |f_n(z) - f(z)| < \epsilon$ for all $z \in D(a_i, r_{a_i})$ Take $N = \max_i N_i$. □

Theorem 5.2. *Let (f_n) be a seq. of holo'c functions on a domain U. Suppose $f_n \to f$ loc. unif. on U, then f is holo'c and $f_n' \to f'$ loc. unif.*

Proof. Preceding theorem implies that $f_n \to f$ unif. on any compact subset. So f is cts (gives integrability). Pick any $a \in U$, and consider $\overline{D(a,r)} \subseteq U$, then by unif. conv. on its closure, have $\int_\gamma f = \lim_n \int_\gamma f_n = 0$ (Cauchy's thm). So f is holo'c on $D(a,r)$ by Morera. So f is holo'c on U. Apply CIF to derivatives

$$|f_n'(w) - f'(w)| = \frac{1}{2\pi}\left|\int_{|z-a|=r} \frac{f_n(z) - f(z)}{(z-w)^2} dz\right|$$

Choose $|w - a| < r/2$ (sufficiently small), then can bound the integral. Then use unif. conv. of (f_n) on compact subsets. □

Proposition 5.3. *Let (f_n) be a seq of holo'c functions on a domain U. Suppose $f_n \to f$ loc. unif. on U. If f_n is injective on U for all n then f is either injective or constant.*

Proof. Suppose non-constant and non-injective, exists $z_1 \neq z_2$ s.t. $f(z_1) = f(z_2) = a$.

By connectedness of U, can construct a (simple) closed curve γ which winds around z_1 once and z_2 once. Since f is non-constant (it takes the value a at most finitely many time in the domain γ bounds), can choose γ so that $f(z) \neq a$ for all $z \in \gamma$. By loc. unif. conv. the same is true for f_n for sufficiently large n. Apply argument principle

$$1 \geq \frac{1}{2\pi i}\int_\gamma \frac{f_n'}{f_n - a} \to \frac{1}{2\pi i}\int_\gamma \frac{f}{f-a} \geq 2$$

contradiction! □

6 Counterexamples

7 Computation Techniques

7.1 Residue computation

1. Simple poles: If $f(z) = g(z)/h(z)$, h has a simple zero at a and g holo'c nonzero at a, then $\mathrm{Res}_{z=a}(f) = g(a)/h'(a)$.
2. Poles of order k: If $f(z) = g(z)/(z-a)^k$, g holo'c and non-zero at a. Then $\mathrm{Res}_{z=a}(f) =$ coeff of $(z-a)^{k-1}$ in g expansion $= g^{(k-1)}(a)/(k-1)!$.
3. In general, need to compute Laurent expansion.

7.2 Basic estimates

7.3 Contour choices

7.4 Basic conformal equivalence

- linear map: rotation and scaling
- Power map: $z \to z^n$, from sectors to sectors/half planes
- Mobius maps: Disk to disk and disk to half plane ($(z-i)/(z+i)$: upper half plane to unit disk) [Can use Mobius maps on any region bounded by circles/lines.]
- Exponential/Log: horizontal strip to sectors/half planes (Some branch of log can be its inverse).