Geo

1 Embedded surfaces

Theorem 1.1 (Implicit function theorem)**.** *Let $T \subseteq \mathbb{R}^{k+l}$ be an open subset (use coordinate (x,z), where $x \in \mathbb{R}^k$, $z \in \mathbb{R}^l$). Let $p = (a,b) \in f^{-1}(0) \subseteq T$ If $\det(\partial_{z_j} \partial f_i) \neq 0$, then there exists open nbd A of a in $\mathbb{R}_k$ and B of b in $\mathbb{R}^l$ and smooth $F : A \to B$ s.t. $A \times B \subseteq T$ and $f^{-1}(0) \cap (A \times B) = \{(x, F(x)) : x \in A\}$.*

Proof. Define $g(x,z) = (x, f(x,z))$. Then compute the derivative, which is invertible, so by IFT Find T, V s.t. $g : T \to V$ is a diffeo with $g(p) = (a, 0)$. Inverse is given by $H(u,v) = (u, H(u,v))$ Restrict to $U = V \cap \mathbb{R}^k \times \{0\}$, get $h(u,0) = (u, F(u))$, where F is a smooth map $U \to \mathbb{R}^l$. Pick A, B (nbd of a, b) sufficiently small s.t. $A \times B \subseteq T$. Shrink A if necessary, may assume that $A \subseteq F^{-1}(B)$, then $F : A \to B$ is the desired smooth map. Can check $(x,z) \in F^{-1}(0)$ iff $g(x,z) = (x,0)$ iff $(x,z) = h(x,0) = (x, F(x))$ iff $z = F(x)$. □

Theorem 1.2 (Equivalent definitions of smoothly embedded surfaces)**.** *Let $\Sigma \subseteq \mathbb{R}^3$. Σ is a smoothly embedded surface if one of the following equivalent conditions holds.*

- *$\forall p \in \Sigma$, $\exists$ open nbd T of p in $\mathbb{R}^3$ and a smooth function $f : T \to \mathbb{R}$ s.t. $\Sigma \cap T = f^{-1}(0)$ and $D_p f$ is non-zeo.*
- *$\forall p \in \Sigma$, $\exists$ open nbd T of p in $\mathbb{R}^3$ and W of 0 and diffeo $g : T \to W$ s.t. $g(\Sigma \cap T) = W \cap (\mathbb{R}^2 \times \{0\})$.*
- *$\forall p \in \Sigma$, there is a parametrization of Σ near p, i.e., there is an open nbd U of p in Σ and open $V \subseteq \mathbb{R}^2$ and a homeomorphism $\sigma : V \to U$ that is smooth as a map to $\mathbb{R}^3$ for all $q \in V$.*
- *Σ is locally a graph over a coordinate plane. (Essentially implicit function theorem)*

Proposition 1.3 (Change of coordinate)**.** *$\sigma(u,v)$ and $\tau(x,y)$ parametrizations, then write $\psi = \sigma^{-1} \circ \tau$, i.e., $\psi(x,y) = (u(x,y), v(x,y))$, then (expand by chain rule)*

$$\begin{pmatrix} \tau_x & \tau_y \end{pmatrix} = \begin{pmatrix} \sigma_u & \sigma_v \end{pmatrix} \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

Definition 1.4 (Gauss map and orientability)**.**

$$\vec{n}(p) = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$$

A smooth surface is orientable if it admits a global Gauss map.

Proposition 1.5. *A smooth surface is orientable iff it can be covered by a collection of parametrizations $\{\sigma_\alpha : \alpha \in I\}$ s.t. all transition functions $\psi_{\alpha\beta} = \sigma_\alpha^{-1} \circ \sigma_\beta$ are orientation preserving, i.e., $\det D\psi_{\alpha\beta} > 0$*

Proof. □

1.1 Fundamental forms

Definition 1.6 (FFF)**.** Family of inner product I_p of tangent space obtained by restricting the standard inner product of $\mathbb{R}^3$.

Proposition 1.7. *In parametrization $\sigma(u,v)$, Can express FFF as a symmetric bilinear form $Edu^2 + 2Fdudv + Gdv^2$, where $E = \sigma_u \cdot \sigma_u$, $F = \sigma_u \cdot \sigma_v$ and $G = \sigma_v \cdot \sigma_v$.*

Proof. Evaluate at basis vectors σ_u and σ_v. □

Definition 1.8 (Pullback)**.** Smooth map $H : \Sigma_1 \to \Sigma_2$ between embedded surfaces. Let $\sigma(u, v)$ be a parametrization of H near p. The pullback of FFF via H is given (in components) by $(H \circ \sigma)_u \cdot (H \circ \sigma)_u = H_u \cdot H_u$... (similarly for v)..

Definition 1.9 (SFF)**.** Let $H : \Sigma \to T_p\Sigma$ be the orthogonal projection, then by direct computation, H is a local diffeo at p so has a smooth inverse H^{-1} defined on an open nbd W of 0 in $T_p\Sigma$. Then there is a unique $f : W \to \mathbb{R}$ s.t. $H^{-1}(w) = p + w + f(w)n(p)$ and $f(0) = 0$, $D_0 f = 0$. Then II_p is the symmetric bilinear form on $T_p\Sigma$ given by the Hessian of f. Given a choice of Gauss map on some $U \subseteq \Sigma$, SFF is the family II_p.

Lemma 1.10. Existence and uniqueness of $f : W \to \mathbb{R}$.

Proof. Define $e : W \to \mathbb{R}^3$ $w \mapsto H^{-1}(w) - (p + w)$. By IFT $D_0 e = 0$. Consider the orthogonal projection $\pi : \mathbb{R}^3 \to T_p\Sigma$, then $\pi(e(w)) = \pi(H^{-1}(w)) - \pi(p) - \pi(w) = w - w = 0$, so $e(w) \perp T_p\Sigma$, so $e(w) = f(w)n(p)$. Clearly $f(0) = 0$ and $D_0 f = 0$ ($D_0 e = 0$). ☐

Proposition 1.11. *Given a parametrization $\sigma(u, v)$, can express SFF as $Ldu^2 + 2Mdudv + Ndv^2$, where $L = \sigma_{uu} \cdot n$, $M = \sigma_{uv} \cdot n$, and $N = \sigma_{vv} \cdot n$.*

Proof. ☐

Definition 1.12 (Gaussian curvature)**.** Fix a Gauss map $n : U \to S^2$, then $D_p n$ is an endomorphism of $T_p\Sigma$. Define (Gaussian curvature) $K(p) = \det D_p n$.

Proposition 1.13. *$D_p n$ is self-adjoint (wrt I_p). Moreover, $I_p(\cdot, D_p n(\cdot)) = -II_p$.*

Proof. Suffices to check the basis vectors of $T_p\Sigma$. Direct computation,

$$\sigma_u \cdot D_p n(\sigma_v) = \sigma_u \cdot (n \circ \sigma)_v = (\sigma_u \cdot n)_v - (\sigma_{uv} \cdot n) = -M$$

Similarly expand $D_p n(\sigma_u) \cdot \sigma_v$ and use commutativity of mixed partial. ☐

Corollary 1.14. $K = \dfrac{LN - M^2}{EG - F^2}$

Proof. Write down the matrices and take det. ☐

1.2 Geodesics

Definition 1.15 (Geodesics)**.** A geodesic in Σ is a path $\gamma : I \to \Sigma$ defined on an open/closed interval s.t. $\forall t \in I$, $\ddot{\gamma}(t) \perp T_{\gamma(t)}\Sigma$ and γ is non-constant.

Theorem 1.16 (Geodesic equations)**.** *A curve $\gamma(t) = \sigma(u(t), v(t))$ is a geodesic iff*

$$\begin{cases} (E\dot{u} + F\dot{v})^\bullet = \frac{1}{2}E_u\dot{u}^2 + F_u\dot{u}\dot{v} + \frac{1}{2}G_u\dot{v}^2 \\ (F\dot{u} + E\dot{v})^\bullet = \frac{1}{2}E_v\dot{u}^2 + F_v\dot{u}\dot{v} + \frac{1}{2}G_v\dot{v}^2 \end{cases}$$

Proof. Expand the condition $\ddot{\gamma} \perp \sigma_u$ and $\ddot{\gamma} \perp \sigma_v$. (Differentiate by MVC). Alternatively, apply E-L equation to the energy functional. ☐

Corollary 1.17. *If γ is a geodesic on Σ, then γ has constant speed, i.e., $\|\dot{\gamma}\|$ is constant.*

Proof. Differentiate, $(\dot{\gamma} \cdot \dot{\gamma})^\bullet = 2\ddot{\gamma} \cdot \dot{\gamma} = 0$ by definition as $\dot{\gamma}(t) \in T_{\gamma(t)}\Sigma$. ☐

Definition 1.18 (Energy)**.** $\gamma : [t_0, t_1] \to \Sigma$. Energy$(\gamma) = \int_{t_0}^{t_1} I_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))dt$

Proposition 1.19. *γ is a geodesic iff it's a staionary point of energy among paths with fixed end-points, i.e., for all one-parameter variation $\Gamma(s, t)$ with $\Gamma(s, t_0) = \gamma(t_0)$ and $\Gamma(s, t_1) = \gamma(t_1)$, $d\mathcal{E}/ds(0) = 0$*

Proof. only if part is easy. IBP on $\mathcal{E}'(0)$ and use the fact that $\ddot{\gamma}(t) \perp T_{\gamma(t)}\Sigma$. Conversely, need to use bump function (probably not required to know) ☐

Proposition 1.20. *γ is a geodesic iff it's a stationary point of length functional and has constant speed.*

Proof. Apply E-L equation to the length functinoal and relate that to the energy functional (square root). □

Proposition 1.21. *Local isometry preserves geodesics*

Proof. (Sheet 3 Q2) □

Theorem 1.22 ((Local) existence and uniqueness)**.**

Proof. Picard-Lindelöf. □

2 Topological and smooth surfaces

Definition 2.1. A topological manifold is a second-countable, Hausdorff, locally Euclidean topological space.

Lemma 2.2. Let X be a topological space that is locally Euclidean. Then

- X is connected iff it's path-connected;
- X is second countable iff it's Lindelöf;
- X is Hausdorff iff it's regular (points and closed sets can be separated by disjoint open sets)

Proof. (Sheet 3 Q5) □

Definition 2.3 (Free and proper group action)**.** An action of a group $G \leq \mathrm{Homeo}(X)$ on a top. space X is free and proper if

- For all $p \in X$, there exists open nbd U of p such that $gU \cap U = \varnothing$ for all $e \neq g \in G$.
- For p_1, p_2 in distinct orbits, there exists open nbd U_i of p_i s.t. $gU_1 \cap gU_2 = \varnothing$ for all $g \in G$.

Proposition 2.4. *If Σ is a top. surface with a free and proper action by $G \leq \mathrm{Homeo}(X)$, then Σ/G is a top. surface (equipped with quotient topology).*

Proof. Important: $q : \Sigma \to \Sigma/G$ is an open map. Check that $q^{-1}(q(T)) = \bigcup_{g\in G} gT$ open for each $T \subseteq \Sigma$ open, then by defn of quotient topology, $q(T)$ is open in Σ/G.

Locally euclidean: Pick a chart of $p \in \Sigma$, say $\varphi' : U' \to V'$. Pick another open nbd U'' s.t. $gU'' \cap U'' = \varnothing$ Let $U = U' \cap U''$. Then prove that $q|_U$ is a homeo (clearly cts and surj, assume not inj, then two distinct pts are related by an element $g \in G$, but by construction $g = e$ is forced, so they are actually the same). Compose, get a chart on Σ/G.

Hausdorff: Use free and properness to choose separating nbd of two points (in distinct orbits) and pass to the orbit space (q is open).

Lindelof: Σ is second countable hence Lindelof, pass to the image (q is cts) and use the fact that Σ/G is locally Euclidean to deduce that Σ/G is also second countable. □

Definition 2.5 (Smooth surface)**.** A smooth surface is a top. surface equipped with a smooth structure (an atlas s.t. transition functions are diffeo)

Orientable if admits a smooth atlas s.t. all transition func ψ satisfy $\det D\psi > 0$.

Definition 2.6 (Smooth map)**.** A map between smooth surfaces is smooth if it's cts (this is crucial) and $\phi_2 \circ F \circ \phi_1^{-1}$ is smooth.

Proposition 2.7. *If a group G acts freely and properly on a smooth surface Σ by diffeo, then Σ/G is a smooth surface.*

Proof. Transition functions are of the form $T = \phi_2 \circ q|_{U_2}^{-1} \circ q|_{U_1} \circ \phi_1^{-1} : W_1 \to W_2$. Pick $p \in W_1$, then $T(p) = gp$ for some g, then Consider any $p' \in W_1 \cap g^{-1}W_2$ (open nbd of p), then $T(p') = g'p$, have $g'p \in W_2$ and $g'p \in g'g^{-1}W_2$, so $W_2 \cap g'g^{-1}W_2 \neq \varnothing$, but W_2 is a subset of the domain of some chart, which is chosen so that it doesn't intersect its non-trivial translation, so $g'g^{-1} = e$. So T acts on $W_1 \cap g^{-1}W_2$ by translation which is smooth, so the transition function is smooth. □

3 Riemannian geometry

Definition 3.1 (Generalization of section 1). • Riemannian metric: smooth family of symmetric bilinear form s.t. it's positive definite at each point.

- Geodesics: a smooth path that satisfies the geodesic equation (equivalently, stationary pt of energy functional under one-parameter variations fixing end pts)
- Pullback (of a Riemannian metric): Let $H : (\Sigma_1, g_1) \to (\Sigma_2, g_2)$ be a smooth map between Riemannian 2-manifolds (surfaces). The pullback $H^* g_2$ is defined in each chart ϕ_1 on Σ_1 as follows. Pick a chart ϕ_2 on Σ_2 then the components A, B, C of the pullback in ϕ_1 is given by

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix}_{\phi_1(p)} = (D_{\phi_1(p)}\psi)^T \begin{pmatrix} E_2 & F_2 \\ F_2 & G_2 \end{pmatrix}_{\phi_2(p)} (D_{\phi_1(p)}\psi)$$

where $\psi = \phi_2 \circ H \circ \phi_1^{-1}$.

Proposition 3.2. *Local isometry iff conformal and area-preserving.*

Proposition 3.3. *If G acts freely and properly by isometry on (Σ, g), then there is a unique Riemannian metric on Σ/G s.t. the quotient map is a local isometry.*

Proof. For any chart $\phi : U \to V$ of Σ, define components of the Riemannian metric on Σ/G on the chart $\phi \circ q|_U^{-1}$ to be the same as g in ϕ. This is the unique choice making q a local isometry. It suffices to prove that this satisfies the transformation law. Start from the transformation law on Σ. Transition on Σ is $\psi = \phi_2 \circ \phi_1^{-1}$ and transition on Σ/G is $\bar{\psi} = (\phi_2 \circ q|_{U_2}^{-1}) \circ (\phi_1 \circ q|_{U_1}^{-1})^{-1}$. They are related by

$$\psi = \bar{\psi} \circ \chi$$

where $\chi := \phi_1 \circ (q|_{U_1}^{-1} \circ q|_{U_2}) \circ \phi_1^{-1}$. The thing in the middle of χ is T, which acts by some elements g so is an isometry by assumption, so apply chain rule and transformation law on Σ, we get the desired result. □

Definition 3.4 (Triangulation). Stuff related to triangulation

- Smooth triangle is a smooth embedding of a closed triangle in $\mathbb{R}^2$ into some Riemannian surface.
- A smooth triangulation of Σ is a collection of smooth triangles covering Σ such that the intersection of two of them is either empty or a common face (edge or vertex).
- A geodesic triangle is a smooth triangle such that the edges are geodesics (up to reparametrization). Can define geodesic polygons similarly.
- A geodesic triangulation is a smooth triangulation such that each smooth triangle is a geodesic triangle.

Theorem 3.5 (Gauss-Bonnet). *For all smooth triangulation of Σ, have $\int_\Sigma K dA = 2\pi\chi(\Sigma)$*

Theorem 3.6 (Gauss-Bonnet for geodesic n-gon). *Let (T, f) be a geodesic n-gon, then $\int_{f(T)} K dA = 2\pi - \sum$ exterior angles*

4 Hyperbolic geometry

Some important groups

- $\mathcal{M}_H$, Mobius maps of the form $\frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$
- $\mathcal{M}_D$, Mobius maps of the form $e^{i\alpha}\frac{z-a}{\bar{a}z-1}$, where $\alpha \in \mathbb{R}$, $|a| < 1$

Definition 4.1. Upper half plane model: equipped with

$$g_H = \frac{dx^2 + dy^2}{y^2} = \frac{|dz|^2}{\text{Im}(z)^2}$$

Definition 4.2. Disk model: equipped with

$$g_D = \frac{4(dx^2+dy^2)}{(1-x^2-y^2)^2} = \frac{4|dz|^2}{(1-|z|^2)^2}$$

This is defined so that the Mobius map $z \mapsto \frac{z-i}{z+i}$ is an isometry from H to D.

Proposition 4.3. *$\mathcal{M}_H$ acts on H by isometry*

Proof. Check generators $z \mapsto az$, $z \mapsto z+c$ and $z \to -1/z$. The first two are straightforward; the last one is almost trivial with complex coord. ☐

Proposition 4.4. *$\mathcal{M}_D$ acts on D by isometry.*

Proof. ☐

Lemma 4.5. If $\phi \in \mathrm{Isom}(D)$, $\phi(0)=0$, and $D_0\phi = \mathrm{id}$, then $\phi = \mathrm{id}$.

Lemma 4.6. If $\phi \in \mathrm{Isom}(H)$, $\phi(i)=i$, and $D_i\phi = \mathrm{id}$, then $\phi = \mathrm{id}$.

Proposition 4.7. *$\mathcal{M}_H = \mathrm{Isom}^+(H)$ and $\mathcal{M}_D = \mathrm{Isom}^+(D)$.*

Proof. Let $\Sigma(u,v) = (u,v)$ be the obvious parametrization.

Clearly $\mathcal{M}_D \subseteq \mathrm{Isom}^+(D)$, Conversely, if $\phi \in \mathrm{Isom}^+(D)$, then by composing with $z \mapsto (z - \phi(0))/(\overline{\phi(0)}z - 1)$ if necessary, can assume WLOG that $\phi(0) = 0$. Now $D_0\phi$ is an automorphism of $\mathbb{R}^2$. Since ϕ is an orientation-preserving isometry, $D_0\phi$ is orthogonal and have determinant $+1$, so by applying a rotation if necessary, we may assume that $D_0\phi$ fixes σ_u, then orientation forces $D_0\phi = \mathrm{id}$, then $\phi = \mathrm{id}$ by preceding lemma, so $\phi \in \mathcal{M}_D$. Done.

Can use similar argument on $D_i\varphi$ for $\varphi \in \mathrm{Isom}^+(H)$. Alternatively conjugate using the map $z \mapsto (z-i)/(z+i)$ and work in D. ☐

Proposition 4.8. *$\mathrm{Isom}(D)$ is generated by $\mathrm{Isom}^+(D)$ and $z \mapsto \bar{z}$; $\mathrm{Isom}(H)$ is generated by $\mathrm{Isom}^+(H)$ and $z \mapsto -\bar{z}$.*

Proof. The last step gives two choices for the image of the second basis vector, so compose with the orientation reversing map if necessary. ☐

5 Calculation

Example 1 (Surface of revolution)**.** *$\sigma(u,v) = (f(u)\cos(v), f(u)\sin(v), g(u))$, then $\sigma_u = (f'(u)\cos(v), f'(u)\sin(v), g'(u))$ and $\sigma_v = (-f(u)\sin(v), f(u)\cos(v), 0)$. So FFF is*

$$(f'^2 + g'^2)du^2 + f^2dv^2$$

Gauss map

$$\vec{n} = \frac{(g'\cos v, g\sin v, -f')}{\sqrt{f'^2+g'^2}}$$

SFF

$$\frac{1}{\sqrt{f'^2+g'^2}}[(f''g' - f'g'')du^2 - fg'dv^2]$$

Gaussian curvature

$$\kappa = \frac{g'(g''f' - g'f'')}{f(f'^2+g'^2)^2}$$

Geodesic equations (assume unit speed parametrization):

Definition 5.1 (Cross ratio)**.**

$$[z_1, z_2; z_3, z_4] = \left(\frac{z_3 - z_1}{z_3 - z_2}\right) \Big/ \left(\frac{z_4 - z_1}{z_4 - z_2}\right) = \frac{(z_3-z_1)(z_4-z_2)}{(z_3-z_2)(z_4-z_1)}$$