

1 Basic stuff about vector spaces

Lemma 1.1 (Steinitz Exchange lemma). Let V be a vector space. Suppose $\{v_1, \dots, v_m\}$ is a linearly independent set and $\{w_1, \dots, w_n\}$ spans V , then

- $m \leq n$
- up to reordering, $\{v_1, \dots, v_m, w_{m+1}, \dots, w_n\}$ spans V

Proof. Proceed by induction. Suppose $\{v_1, \dots, v_l, w_{l+1}, \dots, w_n\}$ spans V , then

$$v_{l+1} = \sum_{j>l} \lambda_j w_j + \sum_{j<l+1} \mu_j v_j$$

By linear independence of $\{v_i\}_{i=1}^m$, there exists at least one $\lambda_j \neq 0$. Up to reordering, assume $\lambda_{l+1} \neq 0$, then $w_{l+1} \in \text{span}(\{v_1, \dots, v_{l+1}, w_{l+2}, \dots, w_n\})$, so $\{v_1, \dots, v_{l+1}, w_{l+2}, \dots, w_n\}$ spans V . The base case is simple.

$$v_1 = \sum_j \lambda_j w_j$$

where $\lambda_1 \neq 0$ up to reordering, so can swap in v_1 .

If $m > n$, then the preceding argument shows that $\{v_1, \dots, v_n\}$ spans V , but then $v_{n+1} \in \text{span}(\{v_1, \dots, v_n\})$ contradicting linear independence. \square

This allows basis extension for vector spaces that has a finite spanning set: take a linear independent set and apply Steinitz exchange lemma to a finite spanning set.

Proposition 1.2 (Inclusion Exclusion of dimension). *Let $U, W \leq V$ be finite dimensional subspaces.*

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$$

Proof. If $U \cap W = \{0\}$, then $U + W = U \oplus W$ and the result follows. Otherwise, take a basis of $U \cap W$. Extend. Prove that the union of the two extended bases form a basis of $U + W$. Count. \square

2 Linear maps and matrices

Definition 2.1 (Elementary row and column operations). Recall elementary matrices. Row operations by left multiplication, column operations by right multiplication.

Theorem 2.2 (Rank-nullity theorem). *Let V be a finite dimensional vector. Let W be a vector space. Suppose $\alpha : V \rightarrow W$ is a linear map, then*

$$\dim V = r(\alpha) + n(\alpha)$$

Proof. Several strategies

- Basis extension (construct a basis for $\ker \alpha$ and extend)
- As a corollary of first isomorphism theorem
- elementary operations (equivalent matrices), slightly complicated when W has infinite dimension.

\square

Theorem 2.3 (Change of basis). B, B' bases of V . C, C' bases of W

$$[\alpha]_{B',C'} = [\text{id}_W]_{C,C'} [\alpha]_{B,C} [\text{id}_V]_{B',B}$$

How to find $[\text{id}_V]_{B,B'}$? Suppose $B = \{v_1, \dots, v_n\}$, $B' = \{v'_1, \dots, v'_n\}$. Have $p_{ij} = ([v'_j]_B)_i$, or in matrix notation, $P = ([v'_1]_B \ [v'_2]_B \ \dots \ [v'_n]_B) = [\text{id}_V]_{B',B}$. If $[v]_{B'} = a_j v'_j$, then $[v]_B = a_j p_{ij} v_i = (p_{ij} a_j) v_i$. This is exactly P applied to $[v]_{B'}$.

Proposition 2.4 (Properties of \det and tr). .

- \det is multiplicative, invariant under similarity transformation;
- tr is linear, and $\text{tr}(AB) = \text{tr}(BA)$, and hence invariant under similarity transformation

We restrict our attention to finite dimensional vector space in the discussion of endomorphisms.

Theorem 2.5 (Triangularization criterion). $\alpha \in L(V, V)$ is triangulable (can be made upper triangular in some basis) iff $\chi_\alpha(t)$ factorizes as (not necessarily distinct) linear factors.

Proof. “only if” is obvious. “if” direction: find an eigenvalue (exist by factorization); construct a basis for the corresponding eigenspace; extend and then α is block upper triangular. Quotient out the eigenspace, and apply strong induction. (Apply properties of \det of block upper triangular matrix to split char polynomial and apply strong induction to the smaller block?) \square

Therefore complex matrices are always triangulable.

Theorem 2.6 (Diagonalization criterion). $\alpha \in L(V, V)$ is diagonalizable iff there exists a polynomial $p(x) \in F[x]$ such that $p(\alpha) = 0$ and $p(x)$ factorizes as distinct linear factors (cannot have double roots...) Over \mathbb{C} , TFAE,

- α is diagonalizable;
- $c_\lambda = 1$ for all eigenvalues λ , where c_λ is the exponent of $(t - \lambda)$ in $m_\alpha(t)$.
- $g_\lambda = a_\lambda$ (geometric multiplicity = algebraic multiplicity) for all eigenvalues.

Proof. If diagonalizable, then take $(x - \lambda_1) \cdots (x - \lambda_k)$, where λ_i are distinct eigenvalues (distinct diagonal entries).

If such polynomial exists, say $p(t) = \prod_{i=1}^k (t - \mu_i) = \sum_i c_i t^i$, then μ_i are distinct eigenvalues. If v is an eigenvector, then $0 = p(\alpha)v = \sum_i c_i \lambda^i v$, so $\lambda = \mu_i$ for some i by considering roots. It's enough to show that $V = \bigoplus_i V_{\mu_i}$ (decompose as the direct sum of eigenspaces).

Define $p_i(t) = \prod_{j \neq i} (t - \mu_j)$, $h_i(t) = \frac{1}{p_i(\mu_i)} \prod_{j \neq i} (t - \mu_j)$, so $h_i(\mu_j) = \delta_{ij}$ and $\sum_i h_i(\mu_j) = 1$ for all j .

But $\sum_i h_i$ is a polynomial of degree at most $k - 1$, so it must be constantly 1.

Define $\pi_i = h_i(\alpha)$, then $\pi_i(V) \leq V_{\mu_i}$ and $\sum_i \pi_i = \text{id}_V$, so $V = \bigoplus_i V_{\mu_i}$, and the result is obvious. \square

Theorem 2.7 (Simultaneous diagonalization). Suppose $\alpha, \beta \in L(V, V)$ are diagonalizable. They are simultaneously diagonalizable iff $\alpha\beta = \beta\alpha$ (they commute).

Proof. If simul diag, then obviously commute. Suppose commute. α is diagonalizable so $V = \bigoplus_\lambda V_\lambda$. If $v \in V_\lambda$, then $\alpha\beta v = \beta\alpha v = \lambda\beta v$, so $\beta(v)$ is an e-vector of α with e-value λ , so $\beta(V_\lambda) \leq V_\lambda$, so it suffices to look at how β acts on each eigenspace V_λ . Since β is diagonalizable, β restricted to each eigenspace V_λ is still diagonalizable. For each λ , choose a basis of V_λ such that $\beta|_{V_\lambda}$ is diagonal. The union is an eigenbasis of α and β , so under this basis α and β are both diagonal. \square

Theorem 2.8 (Cayley-Hamilton theorem). Let V be a finite dimensional, $\alpha \in L(V, V)$ $\chi_\alpha(\alpha) = 0$

Proof. In \mathbb{C} , α is triangulable, so $\chi_\alpha(x)$ is in terms of diagonal entries. Compute $\chi_\alpha(\alpha)(v)$ for all $v \in V$.

In general field, consider $\text{adj}(tI - A)$. Let $B = tI - A$, then $\det B = \chi_A(t) = t^n + a_{n-1}t^{n-1} + \dots + a_1t + a_0$ and $\text{adj}(B) = B_{n-1}t^{n-1} + B_{n-2}t^{n-2} + \dots + B_0$. Have $(tI - A)(B_{n-1}t^{n-1} + \dots + B_0) = \det(B)I = (t^n + \dots + a_0)I$. Match coefficient, and substitute $t = A$, will get a telescoping series which gives 0. \square

Theorem 2.9 (Jordan normal form). c.f. GRM, every matrix can be written in JNF

- (algebraic multiplicity) a_λ = the sum of the sizes of all Jordan block with e-value λ ;
- (geometric multiplicity) g_λ = the number of Jordan blocks with e-value λ
- c_λ = the size of the largest Jordan block with e-value λ .

2.1 Dual spaces

Proposition 2.10 (Canonical embedding). *Let V be a vector space (not necessarily finite dim), then $i : V \hookrightarrow V^{**}, v \mapsto \hat{v}$, where $\hat{v}(\epsilon) = \epsilon(v)$ for $\epsilon \in V^*$.*

For finite dimensional vector space, i is the canonical isomorphism.

3 Bilinear forms

Definition 3.1. $\phi : U \times V \rightarrow F$ (U, V are vector spaces over F) is a bilinear form if it's linear in both argument.

$Q : V \rightarrow F$ is a quadratic form if

1. $Q(\lambda v) = \lambda^2 Q(v)$ for all $\lambda \in F, v \in V$;
2. the map $(u, v) \mapsto Q(u + v) - Q(u) - Q(v)$ is bilinear. (related to polarization identity)
(When $F = \mathbb{C}$, we say $\phi : U \times V \rightarrow \mathbb{C}$ is a sesquilinear form if it's linear in the first argument and conjugate-linear in the second.)
 ϕ is Hermitian if it's sesquilinear and $\phi(u, v) = \phi(v, u)^*$.
The analog of Q satisfies $Q(\lambda v) = |\lambda|^2 Q(v)$.

Proposition 3.2 (Polarization identity). *For symmetric bilinear form ψ , we have*

$$2\psi(u, v) = Q(u + v) - Q(u) - Q(v)$$

(If 2 is invertible in F , then this identity specifies ψ uniquely)

For Hermitian forms

$$\psi(u, v) = \frac{1}{4} [Q(u + v) + iQ(u + iv) - Q(u - v) - iQ(u - iv)]$$

See sheet 4 for a more general version of polarization identity.

Theorem 3.3 (change of basis). *Bilinear forms:*

$$[\phi]_{B'} = P^T [\phi]_B P = [\text{id}_V]_{B', B}^T [\phi]_B [\text{id}_V]_{B', B}$$

For bilinear form $p_{ij} = ([v'_j]_B)_i$, meaning that $v'_j = p_{ij} v_i$ then if $[v]_{B'} = a_j v'_j$, then $[v]_B = a_j p_{ij} v_i$, so

$$\phi(v'_j, v'_m) = \phi(p_{ij} v_i, p_{nm} v_n) = p_{ij} \phi(v_i, v_n) p_{nm}$$

For sesquilinear forms:

$$[\phi]_{B'} = P^T [\phi]_B P^*$$

Same derivation but complex conjugate when pulling things out from the second argument.

Theorem 3.4 (Diagonalization of symmetric bilinear form). *Let V be a finite dimensional vector space of F , and let $\psi : V \times V \rightarrow F$ be a symmetric bilinear form, then there exists a basis B of V s.t. $[\psi]_B$ is diagonal. (Every symmetric matrix is congruent to a diagonal matrix)*

Proof. Proceed by induction on $n = \dim V$. If $\psi(u, u) = 0$ for all $u \in V$, then the matrix is trivial by polarization identity. Otherwise, choose $w \in V$ s.t. $\psi(w, w) \neq 0$. We must have $V = \langle w \rangle \oplus \langle w \rangle^\perp$. To see this, write $v = \lambda w + (v - \lambda w)$, then

$$\psi(w, v - \lambda w) = \psi(w, v) - \lambda \psi(w, w)$$

meaning that we can choose $\lambda \in F$ s.t. RHS vanishes ($\psi(w, w) \neq 0$ by construction so this is possible). Use induction hypothesis to find a basis for $\langle w \rangle^\perp$ s.t. $\psi|_{\langle w \rangle^\perp}$ is diagonal. \square

Over \mathbb{R} , can normalize and reorder so that

$$[\psi]_B = \begin{pmatrix} I_p & & \\ & -I_q & \\ & & 0 \end{pmatrix}$$

Over \mathbb{C} , the result of quadratic form is quite trivial (by including i in normalization, you get an identity submatrix on the top left and 0 elsewhere).

Theorem 3.5 (Diagonalization of Hermitian form). *V finite dim vect space over \mathbb{C} . ψ Hermitian form. There exists a basis where $[\psi]_B$ is*

$$\begin{pmatrix} I_p & & \\ & -I_q & \\ & & 0 \end{pmatrix}$$

The proof is identical to the previous one.

Theorem 3.6 (Sylvester's law of inertia). *Let V be a finite dimensional vector space over \mathbb{R} . Let $\psi : V \times V \rightarrow \mathbb{R}$ be a bilinear form, then the signature of ψ is well-defined.*

If instead V is a finite dimensional vector space over \mathbb{C} and ψ is a Hermitian form, then the analogous notion of signature is also well-defined for $\psi(u, u) = Q(u)$.

Proof. Suppose

$$[\psi]_B = \begin{pmatrix} I_p & & \\ 0 & -I_q & \\ & & 0 \end{pmatrix}, \quad [\psi]_{\tilde{B}} = \begin{pmatrix} I_{\tilde{p}} & & \\ & -I_{\tilde{q}} & \\ & & 0 \end{pmatrix}$$

Let P (resp. N) be the subspace on which $[\psi]_B$ is positive definite (resp. negative semidefinite), \tilde{P} the subspace on which $[\psi]_{\tilde{B}}$ is positive definite, then $\tilde{P} \oplus N \leq V = P \oplus N$, which implies that $\dim P + \dim N = \dim V \geq \dim \tilde{P} + \dim N$, which implies that $p \geq p'$. Reversing the roles of the two matrices shows that $p \leq p'$, so we must have $p = \tilde{p}$. Similar argument shows that $q = \tilde{q}$. \square

4 Inner product space

Theorem 4.1 (Gram-Schmidt). *If we have a sequence $(v_i)_{i \in I}$, $I = \{1, 2, \dots\}$ (which may or may not terminate) of linearly independent vectors in an inner product space V , then there exists a sequence $(e_i)_{i \in I}$ of orthonormal vectors such that $\langle e_1, \dots, e_k \rangle = \langle v_1, \dots, v_k \rangle$ for any $k \in I$.*

Proof. Let $e_1 = v_1 / \|v_1\|$, and

$$e'_{i+1} = v_{i+1} - \sum_{j=1}^i \langle v_{i+1}, e_j \rangle e_j, \quad e_{i+1} = e'_{i+1} / \|e'_{i+1}\|$$

Can show that $e_{i+1} \neq 0$ by linear independence of (v_i) , and $\langle e_1, \dots, e_k \rangle = \langle v_1, \dots, v_k \rangle$ is obvious. \square

Remark 1. Gram-Schmidt can be used to diagonalize quadratic forms (not necessarily positive definite).

Theorem 4.2 (Spectral theorems for self-adjoint operators in finite dimensions). *Let V be an inner product space over \mathbb{R} or \mathbb{C} , and let $\alpha \in L(V, V)$ be a self-adjoint operator, then V has an orthonormal basis of eigenvectors of α .*

Proof. Take a real eigenvalue and an eigenvector v , then $\langle v \rangle \oplus \langle v \rangle^\perp = V$ and $\alpha(\langle v \rangle^\perp) \leq \langle v \rangle^\perp$. To see the latter, note that if $u \perp v$, then $\langle \alpha(u), v \rangle = \langle u, \alpha(v) \rangle = \lambda^* \langle u, v \rangle = 0$. Apply induction hypothesis to find an orthonormal basis of e-vectors in $\langle v \rangle^\perp$ and take union. \square

Theorem 4.3 (Spectral theorems for unitary operators in finite dimensions). *V inner product space over \mathbb{C} . $\alpha \in L(V)$ unitary. V has an orthonormal basis of eigenvectors of α .*

The proof is similar. Just note that eigenvalues have unit modulus and this implies orthogonality of e-vectors with different e-values.

Theorem 4.4 (Spectral theorem for normal operators). *V inner product space over \mathbb{C} If $\alpha \in L(V)$ is normal, then V has an orthonormal basis of eigenvectors of α .*

Proof. Conjugates of eigenvalues of α are precisely eigenvalues of α^* . Eigenvectors corresponding to different e-values are orthogonal. $\text{Im}(\alpha - \lambda I) \perp \ker(\alpha - \lambda I)$. $\ker(\alpha - \lambda I)^2 = \ker(\alpha - \lambda I)$ So JNF is diagonal. Apply Gram-Schmidt to find orthonormal basis for each eigenspace and take union. \square

Proof. Induction: Pick an e-vector v of e-value λ , and consider $V = \langle v \rangle \oplus \langle v \rangle^\perp$. $\alpha(\langle v \rangle^\perp) \leq \langle v \rangle^\perp$ (direct computation). Apply induction hypothesis to find a basis of e-vectors in $\langle v \rangle^\perp$. \square

Theorem 4.5 (Simultaneous diagonalization). *Let V be a finite dim vect space over $F = \mathbb{R}$ (resp. \mathbb{C}). Let $\phi, \psi : V \times V \rightarrow F$ be symmetric bilinear (resp. Hermitian) forms. If ϕ is positive definite then there exists a basis B s.t. $[\phi]_B$ and $[\psi]_B$ are diagonal.*

Proof. There exists a basis such that $[\phi]_C = I$ (by positive definiteness). $[\psi]_C$ is some symmetric (resp. Hermitian) matrix. Use orthogonal (resp. unitary) transformation to diagonalize ψ , then $[\phi]_B$ will still be the identity matrix, which is diagonal. \square

5 Computation techniques

- Complete the square for quadratic forms (Lagrange method). If the quadratic term appears, then go as far as you can (put things in a square), reduce the form to $n - 1$ variables. If only the cross term appear, say $a_{ij}x_i x_j$, $a_{ij} \neq 0$ but $a_{ii} = a_{jj} = 0$, then consider

$$\frac{1}{2a_{ij}} \left(\sum_k (a_{ik} + a_{jk})x_k \right)^2 - \frac{1}{2a_{ij}} \left(\sum_k (a_{ik} - a_{jk})x_k \right)^2$$

This will eliminate both x_i and x_j from the expression thereby reducing the form to $n - 2$ variables. Repeat these two steps until you have completed the square.

- Dual basis. $[\text{id}_V]_{B',B} = P$, then $[\text{id}_{V^*}]_{B'^*,B^*} = [\text{id}_V]_{B,B'}^T = ([\text{id}_V]_{B',B}^{-1})^T$, so the change of basis matrix in the dual basis is $(P^{-1})^T$, so if $([v'_j]_B)_i = P_{ij}$, then $([v'_j]_{B'})_i = (P^{-1})_{ji}$.
- Eigenvalues/Eigenvectors (decomposition as direct sum); generalized eigenspaces. Sheet 2 Q11 is useful: If $\text{id}_V = \sum_i \pi_i$ and $\pi_i \pi_j = 0$ if $i \neq j$, then $V = \bigoplus_i \text{Im}(\pi_i)$.