VP

1 EL equations

Functional:

$$F[y] = \int_a^b f(x, y, y')dx \tag{1}$$

Euler-Lagrange equation (basic):

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

Euler-Lagrange equation (first integral eliminating x):

$$f - y'\frac{\partial f}{\partial y'} = \text{Constant}$$

Euler-Lagrange equation (first integral eliminating y):

$$\frac{\partial f}{\partial y'} = \text{Constant}$$

Euler-Lagrange equation with constraint $G[y] = 0$ (Lagrange multiplier): **Apply EL equation to $F[y] - \lambda G[y]$.**
Euler-Lagrange equation (multiple dependent variables), i.e., $\int_a^b f(x, y_1, \ldots, y_n, y_1', \ldots, y_n')dx$:

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} = 0$$

for each i. (So this is a system of DEs)
First integral eliminating x:

$$f - \sum_i y_i'\frac{\partial f}{\partial y_i'} = \text{Constant}$$

Euler-Lagrange equation (multiple independent variables), e.g., $\int_V f(x_1, \ldots, x_n, y, y_{x_1}, \ldots, y_{x_n})dV$: (summation convention applies)

$$\frac{\partial f}{\partial y} - \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial y_{x_i}} = 0$$

Euler-Lagrange equation (higher derivatives), e.g., $\int_a^b f(x, y, y^{(1)}, \ldots, y^{(n)})dx$:

$$\frac{\partial f}{\partial y} - \sum_{i=1}^{n}(-1)^i\frac{d^i}{dx^i}\frac{\partial f}{\partial y^{(i)}} = 0$$

2 Principles

Fermat's principle:
Least action principle:

3 Legendre Transform

Let $f : S \to \mathbb{R}$, $S \subseteq \mathbb{R}^n$. The Legendre transform is given by

$$f^*(\mathbf{p}) = \sup_{\mathbf{x} \in S} (\mathbf{p} \cdot \mathbf{x} - f(\mathbf{x}))$$

Can take Legendre transform w.r.t one variable.

4 Second Variation

Second variation (basic form): $y \mapsto y + \epsilon\eta$

$$\delta^2 F[y] = \frac{1}{2} \int_a^b \left(\eta^2 \frac{\partial^2 f}{\partial y^2} + 2\eta\eta' \frac{\partial^2 f}{\partial y \partial y'} + \eta'^2 \frac{\partial^2 f}{\partial y'^2} \right) dx$$

Second variation (integrate the middle term by part):

$$\delta^2 F[y] = \frac{1}{2} \int_a^b (Q\eta^2 + P(\eta')^2) dx$$

$$Q = \frac{\partial^2 f}{\partial y^2} - \frac{d}{dx} \frac{\partial^2 f}{\partial y \partial y'}, \qquad P = \frac{\partial^2 f}{\partial y'^2}$$

Theorem 4.1 (Jacobi accessory condition)**.** *If there exists a nowhere vanishing solution to*

$$-(Pu')' + Qu = 0$$

then $\delta^2 F[y] > 0$.

5 Important tricks

Prove global optimum:

- Complete the square (integrand or multivariate functions)
- Consider $F[y + \eta] - F[y]$. Do not assume η is a small variation.

Integration techniques:

1. IBP and sub.
2. Product rule in some way.
3. Add 0.