Algebraic Geometry

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1 Motivating Examples and Introduction

\ : (Owen's signature)

Consider $f(x, y) = x^2 + y^2 - 1$ over $\mathbb{R}$. Roots of f: unit circle.

- 1-dim manifold
- smooth
- irreducible

Algebraically, consider the quotient ring $R[x, y]/I$, where $I = (x^2 + y^2 - 1)$.

- transcendence degree 1 over $\mathbb{R}$.
- localizations are independent of choices
- I is a prime ideal

Example 1.1. $x^n + y^n = z^n$ over $\mathbb{Z}$. (no non-trivial solutions when $n \geq 3$) Assume $n = 2$. Can identify (not a bijective correspondence) the solution sets with $\{(x, y) \in \mathbb{Q}^2 : x^2 + y^2 = 1\}$. Consider the line $L_t : y = tx + 1$. L_t meets the circle at $(-2t/(1 + t^2), (1 - t^2)/(1 + t^2))$. In $\mathbb{Q}^2$, if $t \in \mathbb{Q}$, then

$$t \leftrightarrow \left(\frac{-2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)$$

is a fully algebraic identification of the solution set with the base field.

Remark 1. Care about intersections.

Example 1.2. FTA. The set of zeros of a polynomial over $\mathbb{C}$ is $\{(z, y) = (z, p(z)) : z \in \mathbb{C}\} \cap \{(z, y) : y = 0\} \subseteq \mathbb{C}^2$.

Definition 1.3. Let L/k be a field extension. We say $x \in L$ is algebraic over k if there exists a non-zero $p_x \in k[z]$ s.. $p_x(x) = 0$. Otherwise, x is transcendental over k. Say L/k is algebraic if all elemnts of L are algebraic over k

Recall that every field k has a (unique up to iso) maximal algebraic extension $\bar{k}$, its algebraic closure.

In this course, we work over an algebraically closed field of characteristic 0**.**

1.1 The Projective Plane

FTA predicts that every two lines in a plane intersect at a point, unless they are parallel.

Definition 1.4. The projective plane: $\mathbb{P}^2_k = \mathbb{P}^2 = \{(x, y, z) \in k^3 \setminus \{(0, 0, 0)\}\}/\{(x, y, z) \sim \lambda(x, y, z), \lambda \neq 0\}$.

Denote $[(x, y, z)] = (x : y : z)$. We have an inclusion $k^2 \hookrightarrow \mathbb{P}^2_k, (x, y) \mapsto (x : y : 1)$. The points at infinity are $\{(x : y : 0)\} \subseteq \mathbb{P}^2_k$.

A line $ax + by - c = 0$ in k^2 dosen't have well-defined solution set in $\mathbb{P}^2$, but its homogenization $ax + by + cz = 0$ does.

Definition 1.5. A projective plane curve is

$$\{(x:y:z) \in \mathbb{P}^2 \colon F(x,y,z) = 0\}$$

for some non-zero homogeneous poly F.

If $f \in k[x,y]$, then $z^{\deg f} f(x/z, y/z) = F(x,y,z)$ is homogeneous of the same degree.

Definition 1.6. If $C = \{f(x,y) = 0\} \subseteq k^2$ and F is the homogenization of f, we say that $\{(x:y:z) \in \mathbb{P}^2 : F(x,y,z) = 0\}$ is the projective closure of C in $\mathbb{P}^2$.

2 Affine Varieties

Definition 2.1. Affine n-space over k is the set $\mathbb{A}_k^n = k^n$. A polynomial $f(x_1,...,x_n) \in k[x_1,...,x_n]$ is also a function $\mathbb{A}^n \to k$. The zero set (vanishing locus) of a subset $S \subseteq k[x_1,...,x_n]$ is the set $Z(S) = \{P \in \mathbb{A}^n : \forall f \in S,\ f(P) = 0\}$. An affine algebraic set is any subset of some $\mathbb{A}^n$ of the form $Z(S)$ for some $S \subseteq k[x_1,...,x_n]$.

Definition 2.2. If $f \in k[x_1,...,x_n]$ is a non-constant polynoimal, then $Z(f)$ is a hypersurface. In particular, if f is linear, $Z(f)$ is a hyperplane.

Example 2.3. The twisted cubic is $\{(t,t^2,t^3) \in \mathbb{A}^3 : t \in k\} = Z(t-x^2, z-x^3)$.It is non-planar (not contained in any hyperplane).

Proposition 2.4. *Let $S \subseteq k[x_1,...,x_n]$ be a set of polys. Then*

(i) $Z(S) = Z((S))$, where $(S) \trianglelefteq k[x_1,...,x_n]$ is the ideal generated by S.

(ii) There exists $f_1,...,f_r \in S$ s.t. $Z(S) = Z(f_1,...,f_r)$.

Proof. (i) is trivial. (ii) follows from $k[x_1,...,x_n]$ being Noetherian. □

Proposition 2.5. *Affine algebraic sets satisfy*

(i) $S \subseteq T \subseteq k[x_1,...,x_n] \implies Z(T) \subseteq Z(S)$.

(ii) $\mathbb{A}^n, \varnothing$ are affine algebriac sets.

(iii) Given a collection $\{S_i\}_{i\in I}$ of subsets of $k[x_1,...,x_n]$, $\bigcap_{i\in I} Z(S_i) = Z(\bigcup_{i\in I} S)$.

(iv) If $S,T \subseteq k[x_1,...,x_n]$ are finite, then $Z(S) \cup Z(T) = Z(ST)$.

Proof. (i)-(iii) clear. (iv) by direct calculation. □

Definition 2.6. The Zariski topology on $\mathbb{A}$ is the topology whose closed sets are affine algebraic subsets. This is indeed a topology by preceding proposition.

Definition 2.7. A distinguished open set in $\mathbb{A}^n$ is any set $\mathbb{A}^n \setminus Z(f)$ for a single f.

Note that Zariski topology is very coarse. The intersection of two non-empty open sets is non-empty and dense. Will prove in ES1 that distinguished open sets form a basis of Zariski topology. (Also on ES1) The Zariski topology on a product is the the product of Zariski topology. If $X \in \mathbb{A}^n$ is affine algebraic, then the subspace topology agrees with the Zariski topology on X.

Definition 2.8. A topological space X is irreducible if X is cannot be written as $X = X_1 \cup X_2$ with X_1, X_2 closed and proper. Otherwise, X is reducible.

e.g. $Z(xy)$ is reducible.

Definition 2.9. An affine variety is an irreducible (w.r.t. Zariski topology) affine algebraic set.

If $f \in k[x_1,...,x_n]$ is irreducible then $Z(f)$ is irreducible.

3 Ideals and the Nullstellensatz

Definition 3.1. $X \subseteq \mathbb{A}^n$. The ideal of X is $I(X) = \{f \in k[x_1, ..., x_n] : \forall P \in X,\ f(P) = 0\}$

Proposition 3.2 (Properties of $I(X)$ for X algebraic)**.** *Let X, Y be affine algebraic sets in $\mathbb{A}^n$.*

(1) If $S \subseteq k[x_1, ..., x_n]$, then $S \subseteq I(Z(S))$;

(2) $X = Z(I(X))$

(3) $X = Y$ iff $I(X) = I(Y)$.

(4) $X \subseteq Y$ iff $I(Y) \subseteq I(X)$

Proof. (1) clear from defn.

(2) Clearly $X \subseteq Z(I(X))$. Conversely, write $X = Z(S)$, so $S \subseteq I(X)$, so $Z(I(X)) \subseteq Z(S) = X$.

(3) follows from (2).

(4). If $X \subseteq Y$, then $I(Y) \subseteq I(X)$ by defn. Conversely, if $P \in X \setminus Y$, then (2) implies that $P \notin Z(I(Y))$, so there exists $f \in I(Y)$ with $f(P) \neq 0$. □

Proposition 3.3. *Any affine algebraic set is a finite union (unique up to ordering, ES1) of irred. affine algebraic sets (varieties).*

Proof. Let X be affine algebraic. Suppose X is reducible (otherwise done), i.e., $X = X_1 \cup X_1'$. If X is not a finite union of varieties, then wlog X_1 is not a finite union of varieties. We can write $X_1 = X_2 \cup X_2'$ s.t. X_2 fails to be a finite union of varieties. Continue, get a descending chain of affine alg sets. By the preceding prop, get an ACC in $k[x_1, ..., x_n]$, which eventually stabilizes, i.e., eventually X_n is a finite union of varieties. □

Get maps $Z(\cdot),\ I(\cdot)$

$$\{\text{affine alg subsets of } \mathbb{A}^n\} \overset{X \leftrightarrow I(X)}{\longleftrightarrow} \{I \trianglelefteq k[x_1, ..., x_n]\}$$

$I(\cdot)$ does not have full image, e.g., (x^2).

Proposition 3.4. *$X \subseteq \mathbb{A}^n$ affine alg set. Then X is irreducible iff $I(X)$ is a prime ideal.*

Proof. Suppose X is reducible, write $X = X_1 \cup X_2$ proper closed. Then $I(X) = I(X_1) \cap I(X_2)$. By (3) of the previous proposition, there exists $f \in I(X_1) \setminus I(X_2)$ and $g \in I(X_2) \setminus I(X_1)$. Then $fg \in I(X)$ but $f, g \notin I(X)$, so $I(X)$ is not prime.

Conversely, if $I(X)$ is not prime, then can find $f, g \notin I(X)$ but $fg \in I(X)$. Define $X_1 = X \cap Z(f)$ and $X_2 = X \cap Z(g)$. These are proper closed subsets and $X_1 \cup X_2 = X$, so X is reducible. □

Theorem 3.5 (Weak Nullstellensatz)**.** *The maximal ideals of $k[x_1, ..., x_n]$ are those of the form $(x_1 - a_1, x_2 - a_2, ..., x_n - a_n)$ for some $(a_1, ..., a_n) \in k^n$.*

Proof postponed.

Corollary 3.6 (Weak Nullstellensatz)**.** *If $I \subsetneq k[x_1, ..., x_n]$ is a proper ideal, then $Z(I) \neq \varnothing$.*

Proof. Any proper ideal is contained in a maximal ideal which has the form $(x_1 - a_1, ..., x_n - a_n) = \mathfrak{m}$, so $(a_1, ..., a_n) \in Z(\mathfrak{m})$. □

Definition 3.7. Let $I \trianglelefteq k[x_1, ..., x_n]$. The radical ideal of I is $\sqrt{I} = \{f \in k[x_1, ..., x_n] : \exists m > 0,\ f^m \in I\}$.

Note that $I \subseteq \sqrt{I}$ and $Z(I) = Z(\sqrt{I})$.

Theorem 3.8 (Hilbert's Nullstellensatz)**.** *Let $J \trianglelefteq k[x_1, ..., x_n]$. Then $\sqrt{J} = I(Z(J))$.*

Proof. By defn, $\sqrt{J} \subseteq I(Z(J))$.

Write $J = (f_1, ..., f_r)$ and let $g \in I(Z(J))$. Define another ideal $\tilde{J} = (f_1, ..., f_r, x_{n+1}g(x_1, ..., x_n) - 1) \trianglelefteq k[x_1, ..., x_{n+1}]$. If $\tilde{P} \in Z(\tilde{J})$, then the projection P to the first n coords is in $Z(f_i)$ for all $1 \leq i \leq r$, so $g(P) = 0$. Contradicting $x_{n+1}g - 1 = 0$, so $Z(\tilde{J}) = \varnothing$, so $1 \in \tilde{J}$ by weak Nullstellensatz, so $\exists h_1, ..., h_{n+1} \in k[x_1, ..., x_{r+1}]$ with $\sum_{i=1}^r h_i f_i + h_{r+1}(x_{n+1}g - 1) = 1$. On the set where $x_{n+1}g = 1$, we have $\sum_{i=1}^r h(x_1, ..., x_n, 1/g(x_1, ..., x_n))g(x_1, ..., x_n) = 1$. Clear denominators by a sufficiently high power of g. Get

$$\sum_{i=1}^r h_i'(x_1, ..., x_n) f_i(x_1, ..., x_n) = g(x_1, ..., x_n)^N$$

so $g \in \sqrt{J}$. □

4 Coordinate Rings and Morphisms

Corollary 4.1. *There is a bijective correspondence*

$$\{\textit{affine alg subsets of } \mathbb{A}^n\} \longleftrightarrow \{\textit{radical ideals in } k[x_1, ..., x_n]\}$$
$$X \to I(X)$$
$$Z(J) \leftarrow J$$

Proof. $Z(I(X)) = X$ and $I(Z(J)) = \sqrt{J}$. □

This specializes to

$$\{\text{affine varieties in } \mathbb{A}^n\} \leftrightarrow \{\text{prime ideals of } k[x_1, ..., x_n]\}$$

Definition 4.2. $X \subseteq \mathbb{A}^n$ affine algebraic set. The coordinate ring on X (the ring of regular functions on X) is $A(X) = k[x_1, ..., x_n]/I(X)$. [Alternative notations include $\mathcal{O}_X, \mathcal{O}(X), k[X]$,...]

Remark 2. Have an algebraic description of evaluation of $P \in X$. Write $P = (p_1, ..., p_n)$, then

$$ev_P = k[x_1, ..., x_n] \to k[x_1, ..., x_n]/(x_1 - p_1, ..., x_n - p_n) \overset{\cong}{\to} k$$

Call $\mathfrak{m}_P = (x_1 - p_1, ..., x_n - p_n)$, then $\mathfrak{m}_P = I(\{P\})$. For $P \in X \in \mathbb{A}^n$, $I(X) \subseteq \mathfrak{m}_P$, and the image of $\mathfrak{m}_P$ in $A(X)$ is the ideal of regular functions on X which vanish at P.

Definition 4.3. $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ affine alg. sets. A morphism (regular map) from X to Y is $f : X \to Y$ s.t. $\exists p_1, ..., p_m \in A(X)$ with $f(P) = (p_1(P), ..., p_m(P))$ for all $P \in X$. Denote the set of morphisms $X \to Y$ by $\mathrm{Mor}(X, Y)$. In particular, if $Y = \mathbb{A}^1$, then $\mathrm{Mor}(X, \mathbb{A}^1) = A(X)$.

Definition 4.4. Let $f : X \to Y$ be a morphism. The pullback of f is $f^* : A(Y) \to A(X)$, $g \mapsto g \circ f$.

1) Morphisms are cts w.r.t. Zariski topology. If $Z = Z(g_1, ..., g_r) \subseteq Y$, then

$$f^{-1}(Z) = \bigcap_i f^{-1}(Z(g_i)) = \bigcap_i Z(f^*g) = Z(f^*g_1, ..., f^*g_r)$$

2) Morphisms need not be closed, e.g., $\pi_x : \mathbb{A}^2 \to \mathbb{A}^1$ sends $Z(xy - 1)$ to $\mathbb{A}^1 \setminus \{0\}$.

3) Functorial.

4) Pullback $f^* : A(Y) \to A(X)$ is a ring hom, which restricts to the identity on k, i.e., f^* is a k-algebra hom.

Example 4.5. 1) Let $n \geq m$, $\pi : \mathbb{A}^n \to \mathbb{A}^m$ projection on to the first m coords. Then π is a morphism and $\pi^* : k[y_1, ..., y_m] \to k[x_1, ..., x_n]$ is given by $y_i \to y_i \circ \pi = x_i$, i.e., this map is the natural inclusion.

2) $f : \mathbb{A}^1 \to Z(y - x^2) \subseteq \mathbb{A}^2$, $t \mapsto (t, t^2)$. Then $f^* : k[x, y]/(y - x^2) \to k[t]$, $x \mapsto t,\ y \mapsto t^2$, is an iso. More gnerally, the affine d-Veronese embedding of $\mathbb{A}^1$ is the image of $t \mapsto (t, t^2, ..., t^d) \subseteq \mathbb{A}^d$ (degree d Veronese curve).

Theorem 4.6. *Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ be affine alg. sets. Then $f \mapsto f^*$ defines a bijection* $\mathrm{Mor}(X, Y) \to \mathrm{Hom}_k(A(Y), A(X))$

Proof. Let $x_1, ..., x_n$, $y_1, ..., y_n$ be coords on $\mathbb{A}^n$, $\mathbb{A}^m$ respectively. A morphism $f : X \to Y$ described by $P \mapsto (f_1(P), ..., f_m(P))$, $f_i \in A(X)$. Then $f^*y_i = f_i$ by defn, so f can be recovered from f^*, i.e., the map is inj.

If $\lambda : A(Y) \to A(X)$ is a k-algebra hom. Define $f_i = \lambda(y_i)$ and $f : X \to \mathbb{A}^m$ by $f = (f_1, ..., f_m)$.
Claim: $f(X) \subseteq Y = Z(I(Y))$, i.e., $g \circ f$ vanishes for all $P \in X$ and all $g \in I(Y)$ We have $g \circ f = g(f_1, ..., f_m) = g(\lambda(g_1), ..., \lambda(g_m)) = \lambda(g)$, so this is 0 if $g \in I(Y)$. Note that $\lambda = f^*$, since $\lambda(y_i) = f^*(y_i)$. □

Definition 4.7. A morphism $f : X \to Y$ is an isomorphism if f is bijective and f^{-1} is a morphism.

Note that being bijective does not imply being an iso, e.g., $X = \mathbb{A}^1$, $Y = Z(y^2 - x^3) \subseteq \mathbb{A}^2$. $f : X \to Y$, $t \mapsto (t^2, t^3)$ is a bijective morphism. If f^{-1} is a morphism, then $\exists fg \in k[x, y]$ s.t. $g(t^2, t^3) = t$. Contradiction.

Corollary 4.8. *$f : X \to Y$ is an iso iff f^* is an iso.*

This is affine. Will see that $\hat{\mathbb{C}}$ (Riemann sphere) is a proj. variety. If f is a fn on $\hat{\mathbb{C}}$ which looks like a finite-valued poly everywhere, then it's constant by Liouville.

5 Proof of Nullstellensatz

Definition 5.1. S is a finitely generated R-algebra if $\exists s_1, ..., s_n \in S$ s.t. $S = R[s_1, ..., s_n]$. Say S is integral over R if $\exists$ monic poly $f \in R[x]$ with $f(s) = 0$.

Proposition 5.2. *Let $s \in S$, TFAE,*

1) s is integral over R,

2) $R[s]$ is a f.g. R-mod

3) $\exists$ f.g. R-mod R' which is a subring of S s.t. $R[s] \subseteq R'$.

Proof. 1)$\Rightarrow$2): Take a monic poly f which annihilates s, then $R[s]$ is generated by $1, s, ..., s^{\deg(f)-1}$ as R-mod.

2)$\Rightarrow$3): Take $R' = R[s]$.

3)$\Rightarrow$1): Write $R' = Rv_1 + \cdots + Rv_n$. Consider the multiplication by s. Let A be the matrix of this R-lienar map. Then $Av = sv$, so $(A - s\,\mathrm{id})v = 0$, so $\det(A - s\,\mathrm{id}) = 0$ and $\det(A - s\,\mathrm{id})$ is monic over R, so s is integral over R. ☐

Corollary 5.3. *If $R \subseteq S$ are rings and S is f.g. as an R-alg and R-mod, then all elements of S are integral over R.*

Lemma 5.4 (Zariski's lemma)**.** *Let K/k be a field extension. If K is a f.g. k-algebra, then K is a f.g. k-module.*

Now prove Weak Nullstellensatz assuming Zariski's lemma.

Proof of Weak Nullstellensatz. Note that $(x_1 - a_i, ..., x_n - a_n)$ is maximal in $k[x_1, ..., x_n]$.

Suppose $\mathfrak{m} \trianglelefteq k[x_1, ..., x_n]$ is maximal. so $k \hookrightarrow k[x_1, ..., x_n] \to k[x_1, ..., x_n]/\mathfrak{m} = K$, so K/k is a field extension. K is f.g. as k-algebra, so f.g. as k-mod (Zariski), so the extension is finite. But k is alg-closed, so the extension is trivial. ϕ is surj. Define $a_i = \phi^{-1}(x_i \mod \mathfrak{m})$, so $x_i - a_i \in \mathfrak{m}$. Then $\mathfrak{m} = (x_1 - a_1, ..., x_n - a_n)$ by maximality of the other. ☐

Proposition 5.5. *Suppose $k \subseteq k(s)$ is a field extension and s is transcendental over k. Then*

1) $k[s]$ is a UFD, and contains an infinite set of pairwise coprime elements.

2) If $t \in k(s)$ is integral over $k[s]$, then $t \in k[s]$.

Proof. 1) The map $k[x] \to k[s]$ is an iso. If we can only find a finite set of monic irreducible pairwise coprimes $p_1, ..., p_k$, then $p_1 \cdots p_k + 1$ is pairwise coprime to everything on the list.

2) Write $t = p(s)/q(s)$ and suppose $\exists f_0, ..., f_{n-1} \in k[s]$ s.t. $(p/q)^n + f_{n-1}(p/q)^{n-1} + \cdots + f_0 = 0$. If p, q are coprime, then q is const, i.e., $t \in k[s]$. ☐

Proof of Zariski's lemma. Induction on the number of generators of K as a k-alg. Base case: K is generated by s. $s^{-1} \in k[s]$, so s is alg over k so integral over k, so K is f.g. as a k-mod. Suppose the lemma holds for $< n$ generators. Let $K = k[s_1, ..., s_n]$. Apply inductive hypothesis to $k(s_1)$, so K is a f.g. $K(s_1)$-mod. If s_1 is alg, then done. Suppose s_1 is transcendental over k. Note that each s_i, $2 \leq i \leq n$ is alg over $k(s_1)$, so $\exists f_{ij} \in k(s_1)$ s.t. $s_i^{n_i} + \cdots + f_{i,n-1}s_i^{n_i-1} + \cdots + f_{i,0} = 0$. There exists $f \in k[s_1]$ s.t. $(fs_i)^{n_i} + ff_{i,n-1}(fs_i)^{n-1} + \cdots + f^{n_i}f_0 = 0$, so fs_i is integral over $k[s_1]$. If $h \in k(s_1) \subseteq K = k[x_1, ..., x_n]$, then $\exists N$ s.t. $f^N h \in k[s_1][fs_2, ..., fs_n]$, so $f^N h$ is integral over $k[s_1]$. Since $k[s_1]$ is integrally closed, $f^N h \in k[s_1]$. This holds for all h, so choose $h = 1/t$, t is coprime to f by the previous prop, so contradiction. So s_1 is algebraic. ☐

Definition 5.6. Let $X \subseteq \mathbb{A}^n$ be an affine variety. The function field of X, $k(X)$ is the fraction field of $A(X)$. Its elements are called rational functions on X ($k = \mathbb{C}$: meromorphic). If $\varphi \in k(X)$ can be represented by f/g, with $g(P) \neq 0$ for $P \in X$, then we say φ is regular at P.

Definition 5.7. If $\mathfrak{p}$ is a prime ideal in an ID R with field of fractions K. The localization of R at $\mathfrak{p}$ is $R_{\mathfrak{p}} = \{a/b \in K : b \notin \mathfrak{p}\}$.

Definition 5.8. A local ring is a ring with a unique maximal ideal

Definition 5.9. Let X be an affine variety, and $x \in X$. The local ring of X at x is $\mathcal{O}_{X,x} = \{\varphi = f/g : f, g \in A(X), g(x) \neq 0\}$. This is the localization of $A(X)$ at the prime ideal $\{f \in A(X) : f(x) = 0\}$. (the image of $\mathfrak{m}_x$)

Proposition 5.10. *Let X be an affine variety, $x \in X$. $\mathcal{O}_{X,x}$ has a unique maximal ideal $\mathfrak{m}_x = \{\varphi \in \mathcal{O}_{X,x} : \varphi(x) = 0\}$.*

Proof. $\varphi = f/g \in \mathcal{O}_{X,x}$ has multiplicative inverse iff $f(x) \neq 0$ iff $\varphi \notin \mathfrak{m}_x$. Any proper ideal of $\mathcal{O}_{X,x}$ is contained in $\mathfrak{m}_x$. ☐

Lemma 5.11. *If R is a Neotherian ID and $\mathfrak{p}$ is a prime ideal of R, then $R_\mathfrak{p}$ is Noetherian.*

Proof. Given an ACC in $R_\mathfrak{p}$. Take the preimage under the projection map. Get an ACC in R which stabilizes. Argue by contradiction. ☐

Definition 5.12. X affine variety. For $U \subseteq X$ open, $\mathcal{O}_X(U) = \bigcap_{x \in U} \mathcal{O}_{X,x}$ is the ring of regular functions on U.

Lemma 5.13. $\mathcal{O}_X(X) = A(X)$

Proof. "$\supseteq$" clear.

"$\subseteq$": If $\varphi \in \bigcap_x \mathcal{O}_{X,x}$. Define $I_\varphi = \{h \in A(X) : h\varphi \in A(X)\}$. If I_φ is proper, then it is contained in a maximal ideal. By weak Nullstellensatz, $\exists P \in X$ s.t. $h(P) = 0$ for all $h \in I_\varphi$. But φ is regular at P, i.e., $\varphi = f/g$, $g(P) \neq 0$. Contradiction. So $1 \in I_\varphi$. ☐

Remark 3. It can happen that φ requires more than one representation as fractions, e.g. in $\mathbb{A}^4$, $X = Z(xw - yz)$. Let $\varphi = x/y \in k(X)$ regular on $X \cap \{y = 0\}$. Also $\varphi = z/w$ on X, which is regular for $w \neq 0$, so φ is regular on $(X \cap \{y \neq 0\}) \cup (X \cap \{w \neq 0\})$.

Proposition 5.14. *Let $f : X \to Y$ be a cts (in Zariski topology) map of affine varieties. Then TFAE,*

1) f is a morphism.

2) for all $x \in X$ and all $\varphi \in \mathcal{O}_{Y,f(x)}$, $f^*\varphi \in \mathcal{O}_{X,x}$

Proof. If f is a morphism and $\varphi = g/h$ where $h(f(x)) \neq 0$, then $f^* = (g \circ f)/(h \circ f)$ and $h \circ f \neq 0$ at x, so $f^*\varphi \in \mathcal{O}_{X,x}$.

Conversely, if $\varphi \mapsto \varphi \circ f$ induces a map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ for all $x \in X$, then we have

$$f^* : \bigcap_{f(x) \in f(X) \subseteq Y} \mathcal{O}_{Y,f(x)} \to \bigcap_{x \in X} \mathcal{O}_{X,x} = A(X)$$

Restrict to $A(Y)$, get $f^* : A(Y) \to A(X)$. Evaluating f^* at each coordinate $y_1, ..., y_n$ gives a morphism. ☐

6 Projective Varieties

Consider the projective n-space $\mathbb{P}^n = \mathbb{P}^n_k$. Have $\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}$. Call pts in $\mathbb{P}^{n-1}$ in this decomposition "pts at infinity".

We define $U_i = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : x_i \neq 0\}$ for $0 \leq i \leq n$ (standard affine patch of $\mathbb{P}^n$), and $\mathbb{P}^n \setminus U_i$ is a coordinate hyperplane.

Proposition 6.1. *With the quotient topology, $\mathbb{C}P^n$ is compact.*

Definition 6.2. The Zariski topology on $\mathbb{P}^n$ is the quotient topology induced by the projection $\mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$.

Example 6.3. $n = 1$. (ES1) Zariski closed subsets in k^2 are $\varnothing$, k^2, finite unions of curves and points. So $\varnothing, \mathbb{P}^1$ and the curves which contain lines through 0 are lines through 0 (cf. ES1), so get finite sets too.

Definition 6.4. A polynomial $F \in k[x_0, ..., x_n]$ is homogeneous if all its monomials have the same degree.

Lemma 6.5. *Let F be a homogeneous poly of degree d in $k[x_0, ..., x_n]$. Then $Z(F)$ is a well-defined subset of $\mathbb{P}^n$.*

Proof. $\square$

Definition 6.6. An algebraic set in $\mathbb{P}^n$ is any set of the form $Z(S)$, where S is a collection of homogeneous polys (not necessarily of the same degree).

Can define a topology on $\mathbb{P}^n$ where closed sets are algebraic sets. (Q1ES2) This coincides with the Zariski topology on $\mathbb{P}^n$.

Definition 6.7. An ideal $I \trianglelefteq k[x_0, ..., x_n]$ is homogeneous if it is generated by homogeneous polys (not necessarily of the same degree).

Note that $(x, x^3 + y^2) \trianglelefteq k[x, y]$ is an ideal generated by non-homogeneous polys, but can rewrite it as (x, y^2) which is homogeneous.

Lemma 6.8. *Let $I \trianglelefteq k[x_0, ..., x_n]$. Then TFAE,*

1) I is homogeneous

2) If $f \in I$, then the degree d part $f_d \in I$, where f_d is the sum of all degree d monomials of f.

Proof. 1)⇒2): If I is homogeneous, then $I = (f_1, ..., f_k)$ s.t. $\deg f_i = d_i$ and f_i homogeneous. Given $g \in I$, decompose g as a $k[x_0, ..., x_n]$-linear combination of f_i and write down the degree d part of g.

2)⇒1): If $f_1, ..., f_s$ generate I, then $\bigcup_{i=1}^{s} \bigcup_{d \geq 0} \{(f_i)_d\}$ generate I and are all homogeneous. $\square$

As in the affine case, we have

Proposition 6.9. *For homogeneous ideals $I_j \trianglelefteq k[x_0, ..., x_n]$ we have*

1) $I_1 \subseteq I_2 \implies Z(I_1) \supseteq Z(I_2)$;

2) $\bigcap Z(I_j) = Z(\bigcup I_j)$

3) $Z(I_1) \cup Z(I_2) = Z(I_1 I_2)$.

Definition 6.10. An algebraic set in $\mathbb{P}^n$ is $Z(I)$ for any homogeneous ideal I. A projective variety is any irreducible algebraic set in $\mathbb{P}^n$.

Example 6.11. Suppose $f_1, ..., f_r$ are linear homogeneous in $k[x_0, ..., x_n]$. Each $Z(f_i) \subseteq k^{n+1}$ is a vector subspace, so $Z(f_1, ..., f_r) \subseteq k^{n+1}$ is too. Then $Z(f_1, ..., f_r) \subseteq \mathbb{P}^n$ is a projective linear space in $\mathbb{P}^n$. E.g., $Z(ax + by + cz) \subsetneq \mathbb{P}^2$. This is parametrized by a projective line, for example, if $a \neq 0$ then $(b : a : 0)$ and $(-c : - : a)$ are distinct pts on $Z(ax + by + cz)$. The map $(t, u) \mapsto (bt - cu : -at : au)$ identifies $\mathbb{P}^1$ as this projective linear space.

Example 6.12. Any element of $GL_{n+1}(k)$ induces a well-defined map on $\mathbb{P}^n$. The action has kernel $\{\lambda I : \lambda \in k^*\}$. (cf. ES2)

Varieties do not need to be irreducible.

Definition 6.13. If $F \in k[x_0, ..., x_n]$ is a non-const homogeneous poly, then $Z(F) \subseteq \mathbb{P}^n$ is called a hypersurface in $\mathbb{P}^n$.

Example 6.14. Segre surface $\Sigma_{1,1} = Z(x_0x_3 - x_1x_2) \subseteq \mathbb{P}^3$. Consider the function of sets $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$, $((t_0 : t_1), (u_0 : u_1)) \mapsto (t_0u_0 : t_0u_1 : t_1u_0 : t_1u_1)$. This defines a bijection $\mathbb{P}^1 \times \mathbb{P}^1 \to \Sigma_{1,1}$. This identification is the defn of $\mathbb{P}^1 \times \mathbb{P}^1$ as a projective variety.

Example 6.15. (Rational normal curve of degree n in $\mathbb{P}^n$) Consider $\mathbb{P}^1 \to \mathbb{P}^n$, $(t : u) \mapsto (t^n : t^{n-1}u : \cdots : u^n)$ (bijective to its image). When $n = 3$ this is the (projective) twisted cubic. It's the zero locus of 3 quadrics in $k[x_0, ..., x_3]$ and cannot be written as $Z(f_1, f_2)$ (ES2).

(From projective to affine) Given a proj. variety $V = Z(I) \subseteq \mathbb{P}^n$. Define $I_0 = \{F(1, x_1, ..., x_n) : F \in I\}$ (dehomogenization) in $k[x_1, ..., x_n]$ with $V_0 = Z(I)$. Then $V_0 = C \cap U_0$, where U_0 is the standard patch with $x_0 \neq 0$.

(From affine to proj) If V is an affine variety, then $V \subseteq \mathbb{P}^n$ via an inclusion. This is not necessarily closed in $\mathbb{P}^n$. Can take the closure in Zariski topology.

Definition 6.16. If $f \in k[x_1, .., x_n]$, then the homogenization of f is $F = x_0^{\deg f} f(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})$.

If $V = Z(I) \subseteq \mathbb{A}^n$, then define I^* to be the ideal generated by all homogenization of elements of I. Write $V^* = Z(I^*) \subseteq \mathbb{P}^n$. Under the standard inclusion of the patch U_0, $V^* \cap U_0 = V$. Call V^* the projective closure of V. It is the minimal projective variety containing V, i.e., if $V \subseteq Y = Z((F_1, ..., F_s)) \subseteq \mathbb{P}^n$, then each dehomogenization f_i of F_i vanishes on V, so each f_i is in $I(V)$, so $F_i \in I^*$, so $I(Y) \subseteq I(V^*)$,so $V^* = Z(I(V^*)) \subseteq Z(I(Y)) = Y$.

Example 6.17. The affine Fermat curve $Z(x^n + y^n - 1)$ has homogenization $x^n + y^n = z^n$ which gives a curve in $\mathbb{P}^2$ (the projective closure). The points at infinitey $(z = 0)$ are precisely $(1 : \zeta : 0)$, $\zeta^n = -1$.

Example 6.18. Consider $V = Z(x_0x_2 - x_1^2) \subseteq \mathbb{P}^2$. Dehomogenize w.r.t. each variable, get affine pieces $Z(x_2 - x_1^2)$, $(x_0x_2 - 1)$, $Z(x_0 - x_1^2)$.

Example 6.19. Let F be a homogeneous poly of deg 2 in $k[x_0, ..., x_n]$. $Q = Z(F)$ is a quadric hypersurface. Can choose coordinates so that this is identified with $Z(x_0^2 + \cdots + x_r^2)$, where $r + 1$ is the rank of the quadratic form.

6.1 Projective Nullstellensatz

Definition 6.20. Let $X \subseteq \mathbb{P}^n$ Define the ideal of X to be $I(X)$, generated by homogeneous polys which vanish on X. The affine cone over X is $C(X) = \{(x_0, ..., x_n) \in \mathbb{A}^{n+1} : (x_0 : \cdots : x_n) \in X\} \cup \{0\} \subseteq \mathbb{A}^{n+1}$.

Proposition 6.21 (Projective Nullstellensatz)**.**

(1) If $X_1 \subseteq X_2 \subseteq \mathbb{P}^n$ are algebraic, then $I(X_2) \subseteq I(X_1)$

(2) If $X \subseteq \mathbb{P}^n$ is algebraic, then $Z(I(X)) = X$

(3) For any homogeneous ideal $J \trianglelefteq k[x_0, ..., x_n]$ and $Z(J) \neq \varnothing$, then $I(Z(J)) = \sqrt{J}$.

(4) If J is a homogeneous ideal with $Z(J) = \varnothing$, then either $J = (1)$ or $J(x_0, ..., x_n)$.

Proof. (1) and (2) are the same as affine cases.
(3): Given such J, have $C(Z(J)) \neq \varnothing$. By affine Nullstellensatz, $I(Z(J)) = \sqrt{J}$.
(4): $Z(J)$ viewed in $\mathbb{A}^{n+1}$ is either empty or $\{0\}$. By affine Nullstellensatz, $\sqrt{J}$ contains $x_0, ..., x_n$. □

Definition 6.22. If $V \subseteq \mathbb{P}^n$ is a projective variety and $W \subseteq V$ is also a projective variety, then we say that W is a (closed) subvariety of V, and $V \setminus W$ is an open subvariety of V. Open subvarieties of projective varieties (including $\varnothing$) are quasi-projective varieties.

Proposition 6.23.

(i) Every projective variety is a finite union of irreducible proj. varieties,

(ii) a projective variety X is irreducible iff $I(X)$ is prime.

Proof. (1) identical to the affine argument. (2) If X is reducible, then $\pi : C(X) \setminus \{0\} \to X$ shows that $C(X)$ is reducible, so $I(X)$ is not prime (by the affine version). If $I(X)$ is not prime, so $C(X) = V_1 \cup V_2$, closed proper subsets. Define $Z_i = Z(f(\lambda x_0, ..., \lambda x_n) : f \in I(V_i), \lambda \in k^\times)$ (lines through 0 in V_i). Note that $Z_i \subseteq V_i$, so proper. Since closet subsets of $\mathbb{A}^1$ are $\mathbb{A}^1, \varnothing$, finite sets. Each line is in either Z_1 or Z_2, so $C(X) = Z_1 \cup Z_2$. Apply π, X is reducible. □

A subset $S \subseteq X$ of an algebraic set X is Zariski dense iff any poly f vanishing on S vanishes on X.

Proposition 6.24. *Let $X \subseteq \mathbb{P}^n$ be irred proj. variety, $Y \subsetneq X$ closed subvariety. Then $X \setminus Y$ is Zariski dense in X.*

Proof. Let $f \in I(X \setminus Y)$. By projective Nullstellensatz, $I(X) \subsetneq I(Y)$, so find $g \in I(Y) \setminus I(X)$, so $fg \in I(X)$ and $g \notin I(X)$, so $f \in I(X)$ (X is irred.). □

6.2 Functions on Projective Space

Definition 6.25. If $X \subseteq \mathbb{P}^n$ be an irred projective variety. The function field (field of rational functions) of X is $k(X) = \{f/g : f, g, h \in k[x_0, ..., x_n]$ homogeneous of equal degree, $g \notin I(X)\}/\sim$, where $f_1/g_1 \sim f_2/g_2$ iff $f_1 g_2 - f_2 g_1 \in I(X)$.

Can check that $\sim$ is an equiv relation.

Proposition 6.26. *Let $X \subseteq \mathbb{P}^n$ be a projective variety and $X \subsetneq \{x_0 = 0\}$ (so that $X_0 = X \cap U_0$) is non-empty and open. Then $k(X) = k(X_0)$.*

Proof. The map $f/g \mapsto f(1, y, ..., y_n)/g(1, y, ..., y_n)$ provides an isomorphism. □

Definition 6.27. Let $X \subseteq \mathbb{P}^n$ be an irred projective variety and $\varphi \in k(X)$, and $P \in X$. Then φ is regular at P if $\exists$ representative f/g of φ with $g(P) \neq 0$. Then the local ring at P is $\mathcal{O}_{X,P} = \{\varphi \in k(X) : \varphi$ regular at $P\}$.

Proposition 6.28. *Suppose $X \subseteq \mathbb{P}^n$ not contained in $\{x_0 = 0\}$. Let $P \in X \cap U_0$, where U_0 is the first affine patch. Then $\mathcal{O}_{X,P} = \mathcal{O}_{X_0,P}$.*

Proof. Normalize the first coord, cf. the last proof. □

Example 6.29. If $k = \mathbb{C}$ and consider $\mathbb{P}^1_{\mathbb{C}}$ (Riemann sphere). If $\varphi \in \mathbb{C}(\mathbb{P}^1_{\mathbb{C}})$ is regular at all points, then φ gives a holomorphic map $\mathbb{C}_\infty \to \mathbb{C}$ which is necessarily constant by Liouville.

Proposition 6.30. *There are no non-constant rational functions on $\mathbb{P}^1$ which are regular at all points.*

Proof. If $\varphi \in k(\mathbb{P}^1)$ with representative $f(x_0, x_1)/g(x_0, x_1)$, with no common factors of degree d, then $f(1, x_1), g(1, x_1)$ have no common root. [If $x_1 - \alpha$ is a factor of $g(1, x_1)$, then $x_1 - \alpha x_0$ is a factor of g. If α is a root of g, and $\varphi = h/k$, then $kf - hg = 0$, so $k(1, x_1)f(1, x_1) = h(1, x_1)g(1, x_1)$. Since $f(1, \alpha) \neq 0$, conclude that $k(1, \alpha) = 0$, but now φ fails to be regular at $(1, \alpha)$.] So $g(1, x_1)$ has no roots, so $g(x_0, x_1) = cx_0^d$ for some constant c. Symmetric argument applies to x_0, so $g(x_0, x_1) = c'x_1^d$, so $d = 0$, so φ is constant. □

Corollary 6.31. *$\mathbb{P}^n$ has no non-constant rational functions which are regular everywhere.*

Proof. If $\varphi \in k(\mathbb{P}^n)$ is non-constant, then find $P, Q \in \mathbb{P}^n$ s.t. $\varphi(P) \neq \varphi(Q)$. Restrict to the projective line L connecting P, Q. Then apply the preceding proposition to get a contradiction. □

6.3 Maps Between Projective Varieties

Suppose $F_0, ..., F_m \in k[x_0, ..., x_n]$ are homogeneous of the same degree. Then $F = (F(x_0, ..., x_n), ..., F(x_0, ..., x_n)) : k^{n+1} \to k^{m+1}$. This induces a well-defined map $\mathbb{P}^n \setminus (\bigcap_i Z(F_i)) \to \mathbb{P}^m$. For any homogeneous $G \in k[x_0, ..., x_n]$, $(F_0 G : \cdots : F_m G)$ defines the same map on a smaller set.

Definition 6.32. Let $X \subseteq \mathbb{P}^n$ be a projective variety, $F_0, .., F_m \in k[x_0, ..., x_n]$ homogeneous of the same degree, not all in $I(X)$. Then we have a well-defined map $F : X \setminus \bigcap_i Z(F_i) \to \mathbb{P}^m$. We say that F and G (as above) determine the same rational map if they agree where both defined (a Zariski dense open subset). This is an equiv rel, and an equiv class is called a rational map on X, denoted $F : X \dashrightarrow \mathbb{P}^m$.

[Note that F is equiv to G can be checked by the condition $F_i G_j - G_i F_j \in I(X)$ for all i, j.]

Definition 6.33. A rational map F is regular at $P \in X$ if there exists representative $(F_0 : \cdots : F_m)$ with $F_i(P) \neq 0$. The domain of F is the set of regular points of F_i and $X \setminus$domain is the set of indeterminate points of F.

Definition 6.34. A rational map is said to be a morphism if it is regular at all points of X. In this case write $F : X \to \mathbb{P}^n$ (solid arrow). If $F : X \to Y \subseteq \mathbb{P}^m$ (morphism) for some algebraic subset Y, then F is a morphism. A morphism is an isomoprhism if there exists an inverse. A rational map is birational if it has rational inverse.

Example 6.35. If $F_0, ..., F_m$ are degree 1 homogeneous polys, get a rational map from $\mathbb{P}^n$ to $\mathbb{P}^m$. This is a morphism iff the matrix has full rank $n + 1 \leq m + 1$.

Example 6.36 (Projection from a point)**.** Suppose $P = (0 : \cdots : 0 : 1)$. Define projection from P (to the hyperplane $x_n = 0$) by

$$\pi(x_0 : \cdots : x_n) = (x_0 : \cdots : x_{n-1})$$

Note that π is not regular at P but is regular at all other points. Suppose $X = Z(F)$ for some deg d homogeneous poly. If $P \notin X$, the $\pi_X : X \to \mathbb{P}^{n-1}$ is a morphism. Fix $a = (a_0 : \cdots : a_{n-1}) \in \mathbb{P}^{n-1}$. $\pi^{-1}(a_0 : \cdots : a_{n-1})$ intersects X at $(x_0 : \cdots : x_n) = (a_0\lambda : \cdots : a_{n-1}\lambda : *)$ iff $f(a_0, ..., a_{n-1}, x_n) = 0$. Note that $\deg f(a_0, ..., a_{n-1}, x_n) \leq \deg f$ in the variable x_n, so we have $\leq \deg f$ such points.

Consider $n = 2$, $X = Z(x_0x_2 - x_1^2)$, then $\pi(x_0 : x_1 : x_2) = (x_0 : x_1) \sim (F_0 : F_1)$ iff $x_1F_0 + x_0F_1 \in (x_0x_2 - x_1^2)$, so $F_0 = -x_1$, $F_1 = x_2$ is an example which is defined at P. So $\pi|_X$ is an isomorphism $Z(x_0x_2 - x_1^2) \simeq \mathbb{P}^1$. More generally,

Definition 6.37. Let $n, d \geq 1$ and $N = \binom{n+d}{n} - 1$. Then $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$, $(x_0 : \cdots : x_n) \mapsto (x_0^d : \cdots x_0^{d-1}x_1 : x_0^{d-2}x_2 : \cdots : x_n^d)$ is the degree d Verenoese embedding of $\mathbb{P}^n$.

$\nu_d(\mathbb{P}^n)$ is a projective variety, and ν_d is an isomorphism.

Remark 4. A homogeneous degree d poly F has zero locus $Z(F) \subseteq \mathbb{P}^n \leftrightarrow$ the hyperplane from the coefficients of F intersected with $\nu_d(\mathbb{R}^n)$, i.e., $F = a_0x_0^d + a_1x_0^{d-1}x_1 + \cdots + a_Nx_n^d \leftrightarrow$ hyperplane $a_0z + \cdots + a_Nz_N$

Example 6.38. Let $m, n \geq 1$ and define $N = (n+1)(m+1) - 1$. The Segre embedding is $\sigma_{m,n} : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^N$, $((x_0 : \cdots : x_m), (y_0 : \cdots : y_n)) \mapsto (x_0y_0 : x_0y_1 : \cdots : x_my_n) = (x_iy_j)_{0 \leq i \leq m, 0 \leq j \leq n}$.

Proposition 6.39. *$\sigma_{m,n}$ is a bijection and the projection maps forming its inverse are morphisms. We have $\Sigma_{m,n} := \sigma_{m,n}(\mathbb{P}^m \times \mathbb{P}^n) = Z(I)$, where I is the ideal generated by $(z_{ij}z_{pq} - z_{iq}z_{pj})$, $0 \leq i, p \leq m$, $0 \leq j, q \leq n$. This ideal is prime, so $\Sigma_{m,n}$ is irreducible.*

Proof. $\sigma_{m,n}(\mathbb{P}^m \times \mathbb{P}^n) \subseteq Z(I)$ clear.

If $(a_{00} : \cdots : a_{mn}) \in Z(I)$, then $\exists i, j$ s.t. $a_{ij} \neq 0$ and wlog assme $a_{ij} = 1$. Define $x = (x_0, ..., x_m)$ by $x_p = a_{pj}$ and $y = (y_0 : \cdots y_n)$ by $y_q = a_{iq}$. The image $z_{00} : z_{01} : \cdots z_{mn}$ of (x, y) satisfies $z_{pq} = x_py_q = a_{pj}a_{iq} = a_{pq}$, so the reverse inclusion also holds.

To see that I is prime, note that it is the kernel of $k[z_0, ..., z_{mn}] \to k[x_0, ..., x_m, y_0, ..., y_n]$, $z_{ij} \mapsto x_iy_j$. □

Definition 6.40. Let $X \subseteq \mathbb{P}^m, Y \subseteq \mathbb{P}^n$ be proj. varieties. The Zariski topology on $X \times Y$ is the topology given by identification of $X \times Y$ and $\sigma_{mn}(X \times Y)$.

Example 6.41. $m = n = 1$. $\mathbb{P}^1 \times \mathbb{P}^1$ is identified with $Z(z_0z_3 - z_1z_2)$. Fix $P \in \mathbb{P}^1$, then $P \times \mathbb{P}^1$ and $\mathbb{P}^1 \times P$ are closed curves in $Z(z_0z_3 - z_1z_2)$ isomorphic to $\mathbb{P}^1$.

Theorem 6.42. *Projective varieties are complete, i.e., if X is proj, then for any variety Y, the second projection $X \times Y \to Y$ is closed. (Algebro-geometric version of compactness)*

Proof omitted

Recall that this is false for affine varieties.

Corollary 6.43. *Let $f : X \to Y$ be a morphism of projective varieties. Then f is a closed map.*

Proof. Consider $X \to X \times Y$, $x \mapsto (x, f(x))$. Running through the defn of Segre embedding, $\mathrm{id} \times f$ is a closed map. X is complete, so $f = \pi_2 \circ (\mathrm{id} \times f)$ is closed. □

Corollary 6.44. *Let X be an irreducible projective variety. Then all regular functions on X are constant.*

Proof. If f is regular on X, $f : X \to \mathbb{A}^1 \to \mathbb{P}^1$ is a closed map, so $f(X)$ is a finite union of pts. So $|f(X)| = 1$ by irreducibility, i.e., f is constant. □

6.4 Algebro-geometric Correspondence

Definition 6.45. A rational map $f : X \dashrightarrow Y$ of irreducible projective varieties is dominant if $f(X)$ is dense in Y.

A rational map is birational if it has rational inverse. In this case there exists U open dense in X and V open dense in Y s.t. U is isomorphic to V.

Example 6.46 (Monomial maps)**.** Take $A \in \mathrm{SL}_n(\mathbb{Z})$ and define a rational map $h_A : \mathbb{A}^n \to \mathbb{A}^n$ by $(t_1, ..., t_n) \mapsto (t_1^{a_{11}} \cdots t_n^{a_{1n}}, ..., t_1^{a_{n1}} \cdots t_n^{a_{nn}})$ By defn h_A is a morphism on $\mathbb{A}^n \setminus Z(t_1, ..., t_n)$. Homogenize/extend to a map $\mathbb{P}^n \to \mathbb{P}^n$. This is birational with inverse $h_{A^{-1}}$.

If $A = -I$, get $h_A(t_1, t_2) = (1/t_1, 1/t_2)$. Homogenize. $h_A([t_0 : t_1 : t_2]) = [t_1t_2 : t_0t_2 : t_0t_1]$. [Cremona involution]

Remark 5. If $f : X \to Y$ and $g : Y \to Z$ are rational maps, then $g \circ f$ is a rational map if f is dominant. In particular, if $\varphi : X \to Y$ (rational) is dominant and g is a rational function on Y, then $g \circ \varphi$ is a rational function on X, i.e., a dominant $X \to Y$ induces $\varphi^* k(Y) \to k(X)$ which is injective.

Theorem 6.47. *Let X, Y be varieties. The map $\varphi \mapsto \varphi^*$ is a bijective correspondence*

$$\{\text{dominant rational maps } X \to Y\} \leftrightarrow \{k\text{-extensions } k(Y) \hookrightarrow k(X)\}$$

Proof Sketch (Non-examinable). Wlog, assme X, Y affine. Suppose $i : k(Y) \to k(X)$. Consider generators y_j of $A(Y)$. write $i(y_j) = a_j/b_j$ for $a_j, b_j \in A(X)$. These each define regular functions on $X \setminus \bigcup Z(b_j) = X'$. We obtain a map $A(Y) \to A(X')$, so there is a morphism $X' \to Y$, which is a rational map from X to Y. □

7 Singularities and Tangent Spaces

If $Z(f) \subseteq \mathbb{A}^n$, then there is a linear subspace V at $P = (p_1, ..., p_n) \in X$ given by

$$V = \{(x_1, ..., x_n) \in \mathbb{A}^n : \nabla f \cdot (x - P) = 0$$

This is the "tangent plane" at P

Definition 7.1. Let $X \subseteq \mathbb{A}^n$ be an affine variety. $P \in X$ a point. The tangent space to X at P is

$$T_{X,P} = \{(v_1, ..., v_n) \in k^n : \forall f \in I(X),\ \nabla f(p) \cdot v = 0\}$$

Example 7.2. Consider $X = Z(x + y + z^2 + xyz, x - 2y + z + x^2y^2z^2) \subseteq \mathbb{A}^3$ and $P = (0, 0, 0)$. Then $T_{X,0} = \{(v_1, v_2, v_3) : v_1 + v_2 = 0,\ v_1 - 2v_2 + v_3 = 0\}$ (line)

Example 7.3. $Y = Z(x + y + z^3 + xyz, x + y + x^2 + y^3 + 4z^5)$, $T_{X,0} = \{v_1 + v_2 = 0\}$ (plane)

If $P = 0$, then $T_{X,P}$ is generated by the linear part of elements of $I(X)$.

Suppose $X = \mathbb{A}^n$. If $f(P) = 0$, i.e., $f \in \mathfrak{m}_P$ (either in $k[x_1, ..., x_n]$ or $\mathcal{O}_{X,P}$), define df = linear part of f, i.e., $a_1x_1 + \cdots + a_nx_n$. The map $f \mapsto df$ can be throught of as a map $\mathfrak{m}_P \to T^*_{\mathbb{A}^n,0}$ (dual) The kernel is $\mathfrak{m}_P^2$. There is an identification $T_{\mathbb{A}^n,0} \cong \mathfrak{m}_P/\mathfrak{m}_P^2$ (cotangent space to X at P). [Small algebraic lemma: $\mathfrak{m}_P.\mathfrak{m}_P^2$ is the same when considered in $A(X)$ and $\mathcal{O}_{X,P}$]

Proposition 7.4. *Let $X \subseteq \mathbb{A}^n$ be an affine variety, $P \in X$. Then there is a natural iso of vect spaces $(T_{X,P})^* \cong \mathfrak{m}_P/\mathfrak{m}_P^2$.*

Proof. $X \subseteq \mathbb{A}^n$, so $T_{X,P} \subseteq k^n$. So there is a surj map $(k^n)^* \cong k^n \to T^*_{X,P}$. Wlog, translate to $P = 0$. Composing the restriction map with $f \mapsto df$, we get

$$D : M \to (k^n)^* \to T^*_{X,P}$$

where $M = (x_1, ..., x_n)$. D is surj. Suppose $f \in \ker D$, i.e., $f \in MA(X)$ and $Df = 0$. Then df is a linear combination of linear parts of elements of $I(X)$, so $f \in M^2$, so this descends to the claimed iso. □

Definition 7.5. Let $X \subseteq \mathbb{P}^n$ be a projective variety. $P \in X$. The tangent space to X at P is $T_{X_i,P}$ for any affine patch X_i of X. We write $T_{X,P}$.

The previous proposition, $T_{X,P}$ is well-defined.

If $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ are proj. varieties and $\varphi : X \dashrightarrow Y$ a dominant rational map, $P \in$ domain of φ. Wlog assume affine (patch containing P and patch containing $\varphi(P)$)

Definition 7.6. Write $\varphi = (f_1, ..., f_m)$ Define a linear map $d\varphi_P = T_{X,P} \to T_{Y,\varphi(P)}$ by

$$(v_1, ..., v_n) \mapsto \left(\sum_{i=1}^n v_i \frac{\partial f_1}{\partial x_i}(P), ..., \sum_{i=1}^n v_i \frac{\partial f_m}{\partial x_i}(P) \right)$$

Proposition 7.7. *With the notation above,*

(i) $d\varphi_P(T_{X,P}) \subseteq T_{Y,\varphi(P)}$

(ii) $d\varphi_P$ *is linear and independent of the choice of representation of* φ.

(iii) If $\psi : Y \dashrightarrow Z$ *dominant rational map, then* $d(\psi \circ \varphi)_P = d\psi_{\varphi(P)} \circ d\varphi_P$.

(iv) If φ *is birational, and* φ^{-1} *is regular at* $\varphi(P)$, $d\varphi_P$ *is an iso.*

Proof. (i) By defn, $d\varphi_P(v) \in T_{Y,\varphi(P)}$ iff $\sum_j \sum_i v_i \frac{\partial f_j}{\partial x_i}(P)\frac{\partial g}{\partial y_j}(P) = 0$ for all $g \in I(Y)$. Consider the pullback on function fields. Define $h = g(f_1, ..., f_m) \in I(X)$. Since $(v_1, ..., v_n) \in T_{X,P}$, we have $\sum_i \frac{\partial h}{\partial x_i}(P) = 0$. The result then follows from chain rule.

(ii) linear by defn. Suppose $\varphi = (f_1, ..., f_m) = (f_1', ..., f_m')$, then $f_j - f_j' = p_j/q_j$ where $p_j \in I(X)$ and $q_j(P) \neq 0$,

$$\frac{\partial(f_j - f_j')}{\partial x_i} = \frac{1}{q}\frac{\partial p_j}{\partial x_i}$$

For $v \in T_{X,P}$, $\sum_i v_i \frac{\partial(f_j - f_j')}{\partial x_i} = 0$ for each j.

(iii) Chain rule; (iv) follows from (iii). □

Definition 7.8. Let X be affine or proj. variety.

(i) If X is irreducible, then define $\dim X = \min_{\{P \in X\}} \dim T_{X,P}$.

(ii) If X is irreducible, then say $P \in X$ is non-singular (or smooth) if $\dim T_{X,P} = \dim X$.

(iii) If X is reducible, define $\dim X$ to be the maximal dimension of its irreducible components.

(iv) If X is reducible and $P \in X$ lies on a single irreducible component X_i, then P is nonsingular iff it is nonsingular in X_i. If $P \in X_i \cap X_j$ for distinct irreducible components, then P is singular.

(v) X is smooth if X has no singular points.

(∃− : (Owen's Signature)

Rational maps, function fields, local rings, tangent spaces for proj. varieties require irreducibility.

[$X = Z(xy)$ at 0 shows that we do require irreducibility to have a well-defined tangent space.]

If $X = Z(f) \subseteq \mathbb{A}^2$ and f has not linear terms and $0 \in Z(f)$, then $Z(f)$ is singular at 0, e.g. , the nodal cubic curve $Z(y^2 - x^2(x+1))$ and the cuspidal cubic curve $Z(y^2 - x^3)$.

Remark 6. These curves are not isomorphic, but the tangent spaces don't distinguish them. We need to consider $\mathfrak{m}/\mathfrak{m}^2$, $\mathfrak{m}^2/\mathfrak{m}^3$, ...

Theorem 7.9. *Let* X *be an algebraic variety. Then the set of smooth points of* x *is open dense in* X.

Proof. Since the intersection set is proper closed in X, assume wlog X is irreducible. Also assume X is affine (restrict to a patch if necessary). Say $I(X) = (f_1, ..., f_r)$. If $P \in X$, then $T_{X,P}$ is the zero locus of all linearizations about P, $\dim T_{X,P} = n - \text{rank}(\partial_i f_j)$. $\{P \in X : \text{rank}(\partial_i f_j) \leq n - r\}$ is computed by $(n-r) \times (n-r)$ minors of the matrix $(\partial_i f_j)$ which cannot all be in $I(X)$. □

Corollary 7.10. *If* X, Y *are birational varieties, then* $\dim X = \dim Y$.

Proof. Restrict the birational equivalence $\varphi : X \dashrightarrow Y$ to the smooth locus. □

Definition 7.11. Let L/K be a field extension. We say that $x_1, ..., x_n \in L$ are algebraically independent over K if there is no non-trivial polynomial $p \in k[t_1, ..., t_n]$ s.t. $p(x_1, ..., x_n) = 0$.

Example 7.12. $k(\mathbb{A}^n)$ has n alg indep elements.
$k(Z(y^2 - x(x+1)(x-1))) \cong$ f.f. of $k[x,y]/(y^2 - x(x^2-1))$. Have x, y alg indep, so $\{x\}$ is alg ind set and $\{y\}$ is too, but $\{x, y\}$ is not.

In general if L/K is f.g., generated by $x_1, ..., x_m$, there exists a maximal algebraically independent subset of $\{x_1, ..., x_m\}$. Wlog, assume $\{x_1, ..., x_k\}$ is such a set. So $K \subseteq K(x_1, ..., x_k) \subseteq L$. $K(x_1, ..., x_k)/K$ is purely transcendental; $L/K(x_1, ..., x_k)$ is algebraic. By primitive element theorem, can write $L = K(x_1, ..., x_k)(y)$ for some $y \in L$.

Definition 7.13. Any set $\{x_1, ..., x_r\} \subseteq L$ which generate a pure transcendental extension $K(x_1, ..., x_r)$ s.t. L/K_0 is finite is called a transcendence basis for L/K.

Definition 7.14. The transcendence degree of a f.g. field extension L/K is the size of any transcendence basis.

Proposition 7.15. *Let $K = k(x_1, ..., x_n)$ with $\{x_1, ..., x_n\}$ alg indep over k, and L/K a field extension. If $x_{n+1} \in L$ is algebraic over K, then $I = \{g \in k[t_1, ..., t_n, t] : g(x_1, ..., x_{n+1}) = 0\}$ is a principal ideal generated by an irreducible $f \in k[t_1, ..., t_{n+1}]$. If f contains t_i, then $\{x_1, ..., x_{i-1}, x_{i+1}, ..., x_{n+1}\}$ is an alg indep set over k.*

Proof. Note that $k[t_1, ..., t_n] \subset K$ is suitable for Gauss lemma, and $K[t]$ is a PID. I has a generator in $K[t]$. By Gauss, irred generator. If removing x_i gives an alg dep set, then can find $g \in I$ which doesn't include t_i... □

Corollary 7.16. *Transcendence degree is well-defined*

Proof. NOtes on moodle □

Lemma 7.17 (Reduction to hypersurface). *X irred alg variety. Then X is birational to a hypersurface.*

Proof. Find $K_0 = k(x_1, ..., x_n)$ purely transcendental with $k(X)/K_0$ finite. Write $k(X) = K_0(y)$. The previous proposition says we have irred $f \in k[x_1, ..., x_n][y]$ s.t. $k[x_1, ..., x_n, y] \cong k[x_1, ..., x_n, y]/(f)$, which is the coord ring of a hypersurface. By correspondence, X is birational to hypersurface. □

Corollary 7.18. *For any irred alg variety X, $\dim X$ is the transcendence degree of the function field $k(X)$. Notation: $\mathrm{trdeg}_k\, k(X)$.*

Proof. By reduction lemma, suffices to consider hypersurfaces. If $P \in X$, then $T_{X,P}$ has dimension n or $n - 1$. On a Zariski open subset this dim is $n - 1$. We have $k(X) =$ f.f. of $k[x_1, ..., x_n]/(f)$ for some irred non const f which has transcendence degree $n - 1$, since f is nonconst. □

8 Structure of Algebraic Curves

Definition 8.1. A curve is a smooth projective, irreducible variety of dimension 1.

(unless qualified by singular/affine/reducible)

Example 8.2. Hypersurfaces in $\mathbb{P}^2$, $Z(f)$ for f homogeneous non=const deg d polys in three variables. f is specified by $\binom{d+2}{2}$ coeffs. Using this, can make the following identification:

$$\{\text{deg } d \text{ hypersurfaces in } \mathbb{P}^2\} \leftrightarrow \mathbb{P}^{\binom{d+2}{2}-1}$$

Reducibility of $Z(f)$ is a Zariski closed condition in $\mathbb{P}^{\binom{d+2}{2}-1}$. Note that $(x^d + y^d + z^d)$ is irreducible for every degree (Gauss). Get an open dense set in $\mathbb{P}^{\binom{d+2}{2}-1}$ corresponds to irred hypersurfaces. In $\mathbb{P}^{\binom{d+2}{2}-1} \times \mathbb{P}^2$ there is a closed subvariety defined by $f(P) = 0$. Vanishing of $\partial_x f, \partial_y f, \partial_z f$ are closed conditions since all curves contain smooth points, there is a Zariski closed proper subset in $\mathbb{P}^{\binom{d+2}{2}-1} \times \mathbb{P}^2$ which contains all pairs (f, P) which have $f(P) = 0$ but f is reducible or P is singular for $Z(f)$. A generic hypersurface in $\mathbb{P}^2$ is a curve, e.g., $Z(x^d + y^d + z^d)$ is a curve in $\mathbb{P}^2$.

Remark 7. (Euler's homogeneous identity) If f is homogeneous of deg d in n variables, then $F(x_1, ..., x_n) \deg F = \sum_{i=1}^n x_i \partial_{x_i} F(x_1, ..., x_n)$. This is an identity of polys. (Suffices to check monomials)

Proposition 8.3. *Let C be a curve, $D \subseteq C$ a proper closed subvariety. Then D is a finite union of points.*

Proof. Wlog, C is affine, D is irreducible. Have $I(C) \subsetneq I(D)$. So the inclusion induces a surjective hom $\varphi^* : A(C) \to A(D)$. If D is not a point, then Nullstellensatz implies $A(D) \supsetneq k$. k is alg closed, so $A(D)$ contains a transcendental element (over k). Clear denominators and call the resulting elt t. There exists $x, y \in A(C)$ s.t. $\varphi^*(x) = 1$ and $\varphi^*(y) = t$, so x, y are algebraically indep in $A(C)$ and hence in $k(C)$. Contradiction as $\dim C = 1$. □

Theorem 8.4. *If $P \in C$ is a smooth point of a irred proj. dim 1 (not necessarily smooth) variety, then the ideal $\mathfrak{m}_P \trianglelefteq \mathcal{O}_P$ is a principal ideal.*

Lemma 8.5 (Nakayama's lemma)**.** *Let R be a ring, M a f.g. R-module. Let $J \trianglelefteq R$ be an ideal.*

(i) If $JM = M$, then $\exists r \in J$ s.t. $(1+r)M = 0$.

(ii) If $N \subseteq M$ is a submodule s.t. $JM + N = M$, then $\exists r \in J$ s.t. $(1+r)M \subseteq N$.

Proof. (i): Write $M = y_1 R_1 + \cdots + y_n R$, $y_i \in M = JM$. Write $y_i = \sum_j x_{ij} y_j$, $x_{ij} \in J$. Consider the matrix $X = (x_{ij})$ so that $(I - X)y_i = 0$ for all i. Multiply by adjugate, get $\det(I-X)y_i = 0$ for all i. Note that $\det(I - X) = 1 + r$ for some $r \in J$.

(ii): Apply (i) to M/N and use the correspondence of submodules. □

Proof of thm. Can change coord, so wlog assume $P = 0$ and $T_{C,P} = \{x_2 = \cdots = x_n = 0\}$. Can also assume C is affine. So there exists polys $f_2, ..., f_n \in I(C)$ with $f_j = x_j + h_j$, where temr of h_j all have deg ≥ 2, so in $\mathcal{O}_P$ we have $x_j = h_j \in \mathfrak{m}_P^2$. So $\mathfrak{m}_P = x_1\mathcal{O}_P + \mathfrak{m}_P^2$. Apply Nakayama with $N = x_1\mathcal{O}_P \subseteq M = \mathfrak{m}_P = J \subseteq R = \mathcal{O}_P$. There exists $r \in \mathfrak{m}_P$ s.t. $(1+R)M \subseteq N$. Since $r \in \mathfrak{m}_P$, $1 + r$ is a unit, so $M \subseteq N$, i.e., $\mathfrak{m}_P = x_1\mathcal{O}_P$. □

Corollary 8.6. *Let C is an affine plane curve (possibly singular) $Z(f) \subseteq \mathbb{A}^2$. Let $P \in C$ be a smooth point. Then the function $C \to k$, $Q \mapsto x(Q) - x(P)$ is a local coordinate at P if $\partial_y f(P) \neq 0$.*

Proof. Wlog by translation, assume $P = (0, 0)$ with tangent space $\{aX + bY = 0\}$. THe proof of previous thm shows that $Y \in X\mathcal{O}_p + \mathfrak{m}_P^2$ precisely when $\partial_y f(P) \neq 0$ so in this case X is a local coord at P. □

Theorem 8.7 (Generalized Eisenstein)**.** *Let R be an integral domain, $\mathfrak{p}$ a prime ideal of R, $f(x) = a_n x^n + \cdots + a_1 x + a_0$, $a_n \notin \mathfrak{p}$, $a_i \in \mathfrak{p}$ for $0 \leq i < n$ and $a_0 \notin \mathfrak{p}^2$. Then if $f(x) = g(x)h(x)$ in $R[x]$, g or h is constant.*

Proof. cf. GRM, reduce to $(R/\mathfrak{p})[x]$. □

Example 8.8. $x^d + y^d + z^d \in k[x, y, z]$. Have $y^d + z^d = \prod_{\zeta^d = -1}(y - \zeta z)$. Any $y - \zeta z$ generates a prime ideal. Apply Eisenstein, conclude irred.

Local coord at $P \implies \mathcal{O}_P$ is a discrete valuation ring.

Corollary 8.9. *Let P be a smooth point of a possibly singular curve C. Then there exists a surjective hom $\nu_P : k(C)^\times \to \mathbb{Z}$ called the valuation at P with the following properties [defn of discrete valuation ring]*

(i) $\mathcal{O}_P = \{0\} \cup \{f \in k(C)^* : \nu_P(f) \geq 0\}$

(ii) $\mathfrak{m}_P = \{0\} \cup \{f \in \mathcal{O}_P : \nu_P(f) > 0$

(iii) $\nu_P(x + y) \geq \min\{\nu_P(x), \nu_P(y)\}$ with equality if $\nu_P(x) \neq \nu_P(y)$

If $f \in k(C)^\times$, then for any local parameter π_P at P, we can find a unit $u \in \mathcal{O}_P$ s.t. $f = u\pi_P^{\nu_P(f)}$.

[Often define $\nu_P(0) = \infty$]

Proof. This is a local statement, so assume wlog C is affine. Write $\mathfrak{m}_P = (\pi_P)$ by the previous theorem. Consider the ideal $I = \bigcap_{n \in \mathbb{N}} \mathfrak{m}_P^n$. Clearly have $\mathfrak{m}_P I = I$. I is f.g. as $\mathcal{O}_P$ is Noetherian. Apply Nakayama, find $r \in \mathfrak{m}_P$ s.t. $(1 + r)I = 0$, so $I = 0$ since $1 + r$ is a unit. For any $f \in \mathcal{O}_P \setminus \{0\}$, there exists n s.t. $f \in \mathfrak{m}_P^n \setminus \mathfrak{m}_P^{n-1}$. Define $\nu_P(f) = n$. Given $f = g/h \in k(C)^\times$, choose $g, h \in \mathcal{O}_P$, and define $\nu_P(f) = \nu_P(g) - \nu_P(h)$. Can check well-definedness.

Use $\mathcal{O}_P \supseteq \mathfrak{m}_P \supseteq \mathfrak{m}_P^2 \supseteq \cdots$ to write $g = u\pi_P^{\nu_P(g)}$ and $h = v\pi_P^{\nu_P}$, so $f = uv^{-1}\pi_P^{\nu_P(f)}$. If $x = u\pi_P^m$, $y = v\pi_P^n$, u, v units in $\mathcal{O}_P$ and $n, m \in \mathbb{Z}$, then wlog $m \leq n$ and $x + y = \pi_P^m(u + v\pi_P^{n-m})$. Note that $u + v\pi_P^{n-m}$ is regular at P, so inequality holds. If $n > m$, then $u + v\pi_P^{n-m}$ is a unit, so $\nu_P(x + y) = m = \min\{\nu_P(x), \nu_P(y)\}$. □

Remark 8. If P is a smooth point on a possibly singular curve and $f \in k(C)^\times$, then f or $1/f$ is regular at P.

Theorem 8.10. *Let X be a curve, and $\varphi : X \dashrightarrow \mathbb{P}^m$ a rational map. Then φ is a morphism on X. In particular, birational curves are isomorphic.*

Proof. Wlog choose coords so $\varphi(C) \not\subseteq \{x_0 = 0\}$. Write $\varphi = (G_0 : \cdots : G_m) = (1 : \frac{G_1}{G_0} : \cdots : \frac{G_m}{G_0})$. If $G_i/G_0 \in \mathcal{O}_P$, then φ is regular at P. Suppose not, then define $r = \min\{\nu_P(G_i/G_0 : i = 1, ..., m\}$. Multiply through by π_P^{-t} for some local parameter π_P at P, get $\varphi = (\pi_P^{-t} : \pi_P^{-t}\frac{G_1}{G_0} : \cdots : \pi_P^{-t}\frac{G_m}{G_0})$ which has at least one coord of valuation non-zero, all non-negative, so φ is regular at P.

If $\varphi : X \dashrightarrow Y$ is birational with inverse $\psi : Y \dashrightarrow X$, then $\psi \circ \phi : X \to X$ is birational which is identity on open dense $U \subseteq X$. φ, ψ extend to morphisms, so it suffices to show that if $f : X \to X$ is a morphism with $f|_U = \mathrm{id}_U$ (U open), then $f = \mathrm{id}_X$. The map $f \times \mathrm{id} : X \to X \times X$ maps U to the diagonal which is closed as it's cut out by $x_i y_j - x_j y_i$, where x_i, y_i are the respective coords on the factors of $X \times X$. U is dense in X, so $(f \times \mathrm{id})(X) \subseteq \overline{(f \times \mathrm{id})(U)}$, so $(f \times \mathrm{id})(X) \subseteq \Delta$. □

8.1 Maps Between Curves

Example 8.11. Let $C_d = Z(f_d) \subseteq \mathbb{P}^2$ be a curve defined by a homogeneous poly f_d of deg d. If $P \in \mathbb{P}^2$, then the projection from P gives a morphism $C_d \to \mathbb{P}^1$. Note that smootheness is necessary, e.g. consider the nodal cubic $X = Z(y^2 z - x^2(x+z))$. Has self-intersection at $(0:0:1)$. THen the projection $(x:y:z) \to (x:y) = (1 : y/x)$ canno be ctsly extended to $(0:0:1)$.

Proposition 8.12. *Let $\varphi : X \to Y$ is a non-const morphism of (possibly singular) curves. Then*

(i) $\forall Q \in Y$, $\varphi^{-1}(Q)$ is finite

(ii) $\varphi^* : k(Y) \hookrightarrow k(X)$ is an inclusion, which makes $k(X)$ a finite extension of $\varphi^*(k(Y)) \cong k(Y)$.

Proof. (i): $\varphi^{-1}(Q)$ is closed and φ is nonconst, so finite.

(ii): $\varphi(X)$ is dense in Y, so φ is dominant, so φ^* is inj. If $\in k(Y) \setminus k$, then $x = \varphi^*(t) \in K(X) \setminus k$. Since k is alg closed, x must be transcendental, so $k(X)/k(Y)$ is a finite extension by transcendence degree. □

): Owen is ill today

Definition 8.13. The degree of a map $\varphi : V \to W$ between curves is $\deg \varphi = [k(V) : \varphi^*(k(W))]$.

Definition 8.14. Suppose $P \in V$ and $Q = \varphi(P) \in W$ are smooth points. The ramification degree of φ at P is $e_P = \nu_P(\varphi^* \pi_Q)$, where $\pi_Q \in \mathcal{O}_{W,Q}$ is a local parameter and $\nu_P : k(V)^\times \to \mathbb{Z}$ is the valuation at P. (so $\nu_P(t_p^n) = n$)

Note that $e_P = \min\{\nu_P(x) : x \in \varphi^*(\mathfrak{m}_Q)\mathcal{O}_{V,P}\}$, so e_P doesn't depend on the choice of π_Q.

If φ is an isomorphism, then $\varphi^*(\mathfrak{m}_Q)\mathcal{O}_{V,P} = \mathfrak{m}_P$, so $e_P = 1$, so $e_P = 1$

Definition 8.15. If $e_P = 1$, then we say that φ is unramified at P. Otherwise, say φ is ramified at P.

Picture: $\deg \varphi = \#\varphi^{-1}(Q)$ for generic point $Q \in W$.

Example 8.16. *$E = Z(y^2 - (x^3 - x))$, $\varphi : E \to \mathbb{A}^1$, $(x, y) \mapsto x$. $\varphi^*(x) = x$.* Suppose $P = (x_0, y_0) \in E$ with $\varphi(P) = Q = x_0 \in \mathbb{A}^1$. Then $\varepsilon_P = \nu_P(\varphi^*(x - x_0)) = \nu_p(x - x_0)$. $x - x_0$ is a local parameter of $\mathcal{O}_{E,P}$ iff $\partial_y f \neq 0$ where $f = y^2 - x^3 + x$. So $\nu_P(x - x_0) = 1$ iff $y_0 \neq 0$. At $(\pm 1, 0), (0, 0)$, y is a local parameter. In $\mathcal{O}_{E,P}$, write $y^2 = x^3 - x = (x - x_0)U(x)$ where $U(x) \in k[x]$ doesn't vanish at x_0. Have $\nu_P(y^2) = 2\nu_P(y) = \nu_P(x^3 - x) = \nu_P(x - x_0) + \nu_P(U)$, so $\nu_P(x - x_0) = 2$ for $(\pm 1, 0), (0, 0)$.

Theorem 8.17. *Let $\varphi : V \to W$ be a non-constant map between irreducible projective (possibly singular) curves.*

(i) φ is surjective and for all but finitely many $P \in V$, $e_P = 1$.

(ii) If V and W are smooth, then for all $Q \in W$, $\sum_{P \in \varphi^{-1}(Q)} e_P = \deg \varphi$.

Proof of (i). Since V is projective, completeness theorem shows that $\varphi(V) \subseteq W$ is closed. $\varphi(V)$ is also irreducible and not a point, so $\varphi(V) = W$.

Note $e_P = 1$ iff $\varphi^*(\mathfrak{m}_Q)\mathcal{O}_{V,P} = \mathfrak{m}_P$. WLOG assume affine by intersecting with patches, so $V \subseteq \mathbb{A}^n, W \subseteq \mathbb{A}^m$. Have $\varphi = \pi_W \circ (\mathrm{id} \times \varphi)$. $\mathrm{id} \times \varphi$ is an iso onto its image, so may assume $V \subseteq \mathbb{A}^{n+m}$ and φ is the restriction of projection map $\mathbb{A}^{n+m} \to \mathbb{A}^m$. Suppose $P \in V$ and $Q = \varphi(P) = (q_1, ..., q_m)$, then $\mathfrak{m}_Q = (y_1 - q_1, ..., y_m - q_m)$. So $e_P = 1$ iff $(y_1 - q_1, ..., y_m - q_m)\mathcal{O}_{V,P} = \mathfrak{m}_P$. Apply primitive element theorem to $k(V)/k(W)$. Can write $k(V) \cong k(W)[\alpha]$. Away from the finite set of poles of

α, $\mathfrak{m}_P = (y_1 - q_1, ..., y_m - q_m, \alpha - \alpha(P))$. So $e_P = 1$ iff $\alpha - \alpha(P) \in \mathfrak{m}_Q \mathcal{O}_{V,P}$. α has min poly $p(\vec{y}, x) = x^d + a_{d-1}(\vec{y})x^{d-1} + \cdots + a_0(\vec{y})$, where $a_i(\vec{y}) \in k(W)$. If degree 1, then we are done, so wlog assume $d \geq 2$. By clearing denominators, assume $a_j(\vec{y}) \in k[y_1, ..., y_m]$ Factorize $p(\vec{y}, x) = a_d(\vec{y}) \prod_{i=1}^d (x - \beta_i)$ in $\overline{k(W)}$. Then $\mathrm{disc}(p) = a_d(\vec{y}) \prod_{i \neq j} (\beta_i - \beta_j) = \mathrm{Res}(p, \partial_x p)$ This vanishes identically on W iff p and $\partial_x p$ share a non-trivial factor.

p irred so $\mathrm{Res}(p, \partial_x p) \notin I(W)$, and so $\{Q \in W : \mathrm{Res}(p, \partial_x p)(Q) = 0\}$ is closed proper so finite in W. Since preimages are finite, $U = \{P \in V : \mathrm{Res}(p, \partial_x p)(\varphi(P)) = 0\}$ is open dense in V, so $p$has no repeated root at any pt in U, so $\frac{p(\vec{y},x) - p(\vec{y}, \alpha(P))}{\alpha - \alpha(P)} \in k(V)$ is a unit in $\mathcal{O}_{V,P}$ for all $P \in U$. Also have $p(\vec{y}, x) \in I(V)$ by defn, and $p(\vec{y}, \alpha(P)) = p(y, \alpha(P)) - p(\varphi(P), \alpha(P)) \in \mathfrak{m}_{\varphi(P)}$, so $\alpha - \alpha(P)$ is a unit times an element of $\mathfrak{m}_{\varphi(P)}$. □

Remark 9. If remove the set $\{Q \in W : a_d(Q) = 0\}$, then Q has exactly d preimages in V under φ, all unramified, and so for the ramining points $Q \in W$, we have $\sum_{\varphi(P)=Q} e_P = \sum_{p_1,...,p_d} = 1 = d = \deg \varphi$.

Corollary 8.18. *Let V be a curve, and $f \in k(V)^\times$ is a non-zero rational function.*

(i) If f is regular on V, then f is constant.

(ii) The set of points P with $\nu_P(f) \neq 0$is finite, and $\sum_{P \in V} \nu_P(f) = 0$.

Proof. Use $\mathbb{A}^1 \to \mathbb{P}^1, t \mapsto (1 : t)$. View f as a morphism $\varphi : V \to \mathbb{P}^1$. For (i), f is regular at P iff $\varphi(P) \neq (0 : 1)$, so f regular on $V \implies \varphi$ not surj, so const.

For (ii) $\nu_P(f) \neq 0$ iff f vanishes at P or $1/f$ vanishes at P. $\varphi^{-1}(0)$, $\varphi^{-1}(\infty)$ are finite by hypothesis. If f is const then $\nu_P(f)$ for all $P \in W$. If f is nonconst, then so is φ. Let $t = x_1/x_0$ be a local coord at $(1 : 0)$, and φ^* and $\varphi^* t = f$, so $e_P = \nu_P(\varphi^* t) = \nu_P(f)$. Similarly at ∞ $1/t = x_0/x_1$ is a local coord. $\varphi(P) = \infty \implies e_P = \nu_P(\varphi^*(1/t)) = \nu_P(1/f) = -\nu_P(f)$, so $\sum_{p \in V} \nu_P(f) = \sum_{\varphi(P)=0} \nu_P(f) + \sum_{\varphi(P)=\infty} \nu_P(f) = \sum_{\varphi(P)} e_P - \sum_{\varphi(P)=\infty} e_P = 0$ by previous thm. □

8.2 Divisors on Curves

Definition 8.19. A divisor on a curve V is a finite formal sum $\sum_{P \in V} n_P P$, $n_P \in \mathbb{Z}$. The set of divisors is the abelian group $\mathrm{Div}(V) = \bigoplus_{P \in V} \mathbb{Z}\langle P \rangle$. The degree $\sum n_P P$ is $\sum n_P$.

$D = \sum n_P P \mapsto \sum n_P$ is a hom, with kernel $\mathrm{Div}^0(V)$. The set $\{P \in V : n_P \neq 0\}$ is the support of the divisor D.

Definition 8.20. Let $f \in k(V)^\times$ be a non-zero rational function. The divisor of f is $\mathrm{div}(f) = \sum_{P \in V} \nu_P(f) P$. [By previous corollary, $\deg(\mathrm{div}(f)) = 0$.] Any $\mathrm{div}(f)$ is a principal divisor on V; the principal divisor form a subgroup of $\mathrm{Div}(V)$ (resp. $\mathrm{Div}^0(V)$). The quotient $\mathrm{Div}(V)/\{\text{principal divisors}\}$ is the divisor class group of V, $\mathrm{Cl}(V)$ (or Picard group), (resp. degree 0 divisor class group of V, $\mathrm{Cl}^0(V)$).

Definition 8.21. D is linearly equivalent to D' if there exists $f \in k(V)^\times$ s.t. $D - D' = \mathrm{div}(f)$. Write $D \sim D'$.

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If $f, g \in k(V)$ on a curve V, then $\mathrm{div}(f) = \mathrm{div}(g)$, then $\nu_P(f) = \nu_P(g)$ for all $P \in V$, so $\nu_P(f/g) = 0$ for all P, so $\mathrm{div}(f/g) = 0$, so f/g is constant since it doesn't give a surje morphism $V \to \mathbb{P}^1$.

Proposition 8.22. *Any degree 0 divisor on $\mathbb{P}^1$ is principal, so $\mathrm{Cl}^0(\mathbb{P}^1) = \{0\}$ and $\mathrm{Cl}(\mathbb{P}^1) \cong \mathbb{Z}$.*

Proof. Suppose D is a degree 0 divisor on $\mathbb{P}^1$. Write $D = \sum n_P P = n_\infty \infty + \sum_{p \in \mathbb{A}^1} n_P P$. Note that $n_\infty = -\sum_{p \in \mathbb{A}^1} n_P$. Construct the poly $f(t) = \prod_{p \in \mathbb{A}^1} (t - P)^{n_P}$, $t = x_1/x_0$. Since $1/t$ is a local coord at $\infty = (0 : 1)$, $\nu_\infty(f) = -\sum_{p \in \mathbb{A}^1} n_P = n_\infty$ as needed □

Remark 10. If V is a curve $\mathrm{Cl}^0(V) = \{0\}$, then for $P \neq Q$ on V, $P - Q = \mathrm{div}(f)$ for some $f \in k(V)$. f determines a morphism $\varphi : V \to \mathbb{P}^1$ with $\varphi^{-1}(0) = P, e_P = 1$, so $\deg \varphi = 1$ so $k(V) = k(\mathbb{P}^1)$, so $V \simeq \mathbb{P}^1$.

Definition 8.23. $V \subseteq \mathbb{P}^n$ a curve and consider a hyperplane $H = Z(L) \subseteq \mathbb{P}^n$ not containing V. The hyperplane section of V by H is the divisor $\mathrm{div}(H) = \sum n_P P$, where $n_P = \nu_P(L/x_i)$ for any $x_i(P) \neq 0$.

Remark 11.

1. $x_i(P), x_j(P) \neq 0 \implies x_i/x_j \in \mathcal{O}_P^\times \implies \nu_P(L/x_i) = \nu_P(L/x_j)$ (well-defined)
2. All coeffs $n_P \geq 0$ for hyperplane section
3. If $H = Z(L), H' = Z(L')$ are two hyperplanes not containing V, then $\text{div}(H') - \text{div}(H) = \text{div}(L'/L)$, so all hyperplane sections have the same divisor class. Call this the hyperplane class

Definition 8.24. The degree of $V \subseteq \mathbb{P}^n$ (curve) is $\deg(V) := \deg(\text{div}(H))$ for any hyperplane H not containing V.

If $f : V \overset{\simeq}{\to} W$ is an isomorphism of curves, then $\text{Div}(V) \cong \text{Div}(W)$. Principal divisors are identified so $\text{Cl}(V) \cong \text{Cl}(W)$. However, degree need not be preserved, e.g., the twisted cubic in $\mathbb{P}^3$ has degree 3, but it's isomorphic to $\mathbb{P}^1$. (cf. Veronese embedding) (linear) Change of coordinate on $\mathbb{P}^n$ preserves degree.

If G is homogeneous of degree m, $V \subseteq \mathbb{P}^n$ curve, then can define $\text{div}(G) = \text{div}(Z(G)) = \sum n_P P$, where $n_P = \nu_P(G/x_i^m)$, $x_i(P) \neq 0$, so $\text{div}(G) \sim m\,\text{div}(H)$.

Example 8.25. Let $V \subseteq \mathbb{P}^2$ be a curve. Claim that there exists a line not tangent to V at any point. Recall can identify the space of lines $\{ax + by + cz = 0\}$ in $\mathbb{P}^2$ with points $(a : b : c)$. Consider the map $P \mapsto$ line tangent to V at P is a morphism $V \to \mathbb{P}^2$, which has image dimension ≤ 1, not surj.

Apply a linear change of coords to assume $\{x_0 = 0\}$ is such a line. Write $V = Z(f)$, where f is homogeneous of degree d. We have $f(0, x_1, x_2)$ is a deg d homogeneous poly in 2 variables with no repeated linear factors, so there exists exactly d pts in $V \cap H$, so $\deg(V) = d$.

Definition 8.26. We say a divisor $D = \sum m_P P$ is effective if $n_P \geq 0$ for all P.

Definition 8.27. Let $D = \sum n_P P$ be a divisor on a curve V. The space of rational functions with poles bounded by D is $L(D) = \{f \in k(V) : \nu_P(f) + n_P \geq 0\} = \{f \in k(V) : f = 0 \text{ or } \text{div}(f) + D \text{ is effective}\}$.

The map $f \mapsto \text{div}(f) + D$ identifies $L(D)$ with the set of effective divisors equivalent to D.
The inequality $\nu_P(f + g) \geq \min\{\nu_P(f), \nu_P(g)\}$ implies that $L(D)$ is a vector space over k.

- If $P \notin \text{supp}(D)$, this requires f regular at P
- If $n_P > 0$, f has a pole of order $\leq n_P$ at P
- $n_P < 0$, f has a zero of order $\geq -n_P$ at P

Example 8.28. $V = \mathbb{P}^1$. Write $(0 : 1) = \infty$. Let $D = m(\infty)$, $m > 0$. Write $x = x_1/x_0$. See that $L(D)$ is spanned by $1, x, x^2, ..., x^m$, so $L(D)$ picks out polys of deg $\leq m$, and $\dim L(D) = m + 1$.

Definition 8.29. $\ell(D) := \dim L(D)$

Proposition 8.30. *Let D be a divisor on a curve V.*

(i) $\deg D < 0 \implies L(D) = \{0\}$

(ii) $\deg(D) \geq 0 \implies \ell(D) \leq \deg(D) + 1$

(iii) $\ell(D) \leq \ell(D - P) + 1$ *for all* $P \in V$

Proof. (i) If $\deg(D) < 0$, then no effective divisor can be equiv to D. (iii) implies (ii) by induction on $\deg D$. (iii) Have $\text{ev}_P : L(D) \to k$, $f \mapsto (\pi_P^{n_P} f)(P)$ for any local parameter π_P at P. This is a well-defined homomorphism, with kernel $\{f : \nu_P(f) + n_P \geq 1\} = \{f : \nu_P(f) + n_P - 1 \geq 0\} = \ell(D - P)$. The image has dimension 1, so $\ell(D - P) \geq \ell(D) - 1$. □

Proposition 8.31. *Suppose $D, E \in \text{Div}(V)$, $D \sim E$. Then $L(D) \cong L(E)$ and $\ell(D) = \ell(E)$.*

Proof. $D - E = \text{div}(g)$ for some $g \in k(V)$. Consider $L(D) \to L(E)$, $f \mapsto fg$. $f \in L(D)$ iff $f \in L(E + \text{div}(g))$ iff $\text{div}(f) + \text{div}(g) + E \geq 0$ iff $\text{div}(fg) + E \geq 0$ iff $fg \in L(E)$. □

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Theorem 8.32 (Weak Bezout for curves)**.** *Any two distinct plane curves intersect. If the degreeso f curves are m, n, then they intersect in $\leq mn$ pts.*

Proof. Let F, G be homogeneous in 3 variables, degree m, n resp., call $E = Z(F), D = Z(G) \subseteq \mathbb{P}^2$. Can find an effective divisor $\text{div}(G)$ which has the form $\sum_{P \in C \cap D} n_P P$. WTS $1 \leq \deg(\text{div}(G)) \leq mn$. We know $\text{div}(G) \sim \text{div}(L^n)$ for any lienar form L with $C \not\subseteq Z(L)$. Suffices to show that $1 \leq \text{div}(L) \leq m$. From ES1, $\exists P \in C \cap L$ (L intersects V transversely) s.t. $\text{div}(L) = \sum_{P \in C \cap Z(L)} P$. Change coords so $L = x_0$, algebraic closure $\implies 1 \leq \text{div}(L) \leq m$. □

Remark 12. For general choice of F, G, $|C \cap D| = mn$. With correct notion of multiplicity $I(C \cap D, P)$, have $\sum_{P \in C} I(C \cap D, P) = mn$. $I(C \cap D, P) = \nu_P(G/x_P^n)$ in $\mathcal{O}_{C,P} = \dim_k \mathcal{O}_{\mathbb{A}^2,P}/(f,g)$, where f, g are dehomogenization of F, G in coords with $P \in \mathbb{A}^2$.

Definition 8.33. Let $X \subseteq \mathbb{P}^n$ be a projective variety of dim m. The degree of $X \hookrightarrow \mathbb{P}^n$ is $\#(X \cap H_1 \cap \cdots \cap H_m)$ for generic hyperplanes $H_1, ..., H_m$

Theorem 8.34 (Bezout for hypersurfaces)**.** *Let $X \subseteq \mathbb{P}^n$ be a projective variety, $F \in k[x_1, ..., x_n]$ homogeneous poly with $Z(F)$ not containing any componenet of X. Then* $\deg(X \cap Z(F)) = \deg(X)\deg(F)$.

Proof omitted.

Corollary 8.35. *There is no nonconst morphism $\mathbb{P}^2 \to \mathbb{P}^1$*

Proof in ES3.

Theorem 8.36. *If $f : \mathbb{P}^n \to \mathbb{P}^n$ is an iso, then $f(x) = Ax$ for some $A \in GL_{n+1}(k)$.*

Proof. H hyperplane in $\mathbb{P}^n$, L line, $L \not\subseteq H$. Then $\text{div}(H)$ on L is a pt with coeff 1, so has degree 1, so $\text{div}(f(H))$ on $f(L)$ has deg 1. $f(H)$ and $f(L)$ have degree 1, so $f(H)$ is a hyperplane. Apply to coord hyperplanes $\{x_i = 0\} = H$ get $f(H) = \{a_{0i}x_0 + \cdots + a_{ni}x_n = 0\}$. Let $A = (a_{ji})$. □

8.3 Differentials

K/L field extension. A differential is a k-linear combination of formal symbols xdy with $x, y \in K$ s.t.

(1) $d(\cdot)$ is linear.

(2) Leibniz rule $d(xy)$

(3) $da = 0$ for $a \in k$.

Definition 8.37. The space of differentials $\Omega_{K/k} = \Omega_K$ is the quotient M/N where $M = \langle \delta x : x \in K \rangle$ and N is the subspace generated by

(1) $\delta(x + y) = \delta x - \delta y$;

(2) $\delta(xy) = x\delta y - y\delta x$

(3) δa, $a \in k$

For $x \in K$, define dx to be the coset $\delta x + N$. The map $d : K \to \Omega_{K/k}$ is the exterior derivative.

Lemma 8.38. *Suppose K/k is f.g. with a transcendence basis $\{x_1, ..., x_n\}$. Then if $f \in k(x_1, ..., x_n)$ is a rational function in $x_1, ..., x_n$ and $y = f(x_1, ..., x_n)$.*

$$dy = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_1, ..., x_n)dx_i$$

$\{dx_1, ..., dx_n\}$ is a basis for Ω_K.

Definition 8.39. Let V be a curve, $K = k(V)$. We write $\Omega_V = \Omega_{K/k}$, the space of rational differentials on V. A differential $\omega \in \Omega_V$ is regular at $P \in V$ if we can express $\omega = \sum_i f_i dg_i$, f_i, g_i are regular at P.

Definition 8.40. $\Omega_{V,P} = \{\omega \in \Omega_V : \omega \text{ regular at } P\}$

Proposition 8.41. $\Omega_{V,P} = \mathcal{O}_{V,P} d\pi_P$ *for any local parameter* π_P.

Proof. $\supseteq$: clear from defn. Partials of g/h regular at P iff $h(P) \neq 0$. dx_i generate $\Omega_{V,P}$ over $\mathcal{O}_{V,P}$, so $\Omega_{V,P}$ is finitely generated over $\mathcal{O}_{V,P}$.

If $f \in \mathcal{O}_{V,P}$, $d(f - f(P)) = d(\pi_P g) = g d\pi_P + \pi_P dg \in \mathcal{O}_{V,P} d\pi_P + \pi_P \Omega(V, P)$, for some $g \in \mathcal{O}_{V,P}$. By Nakayama ($R = \mathcal{O}_{V,P}, M = \Omega_{V,P}, N = \mathcal{O}_{V,P} d\pi_P, J = \mathfrak{m}_P$), so $M \subseteq N$. □

If π_P is a local parameter at P, then π_P is transcendental over k, so $d\pi_P$ generates $\Omega_V^\times$ so any differential $\omega = h d\pi_P$. h depends on the choice of π_P, but $\nu_P(h)$ is indep of π_P.

Definition 8.42. In the above setting, define $\nu_P(\omega) = \nu_P(h)$ for any choice of local parameter π_P. If $\nu_P(\omega) = 0$ then ω has a zero at P. If $\nu_P(\omega) < 0$, then ω has a pole at P. If $\omega \in \Omega_V^\times$ then $\operatorname{div}(\omega) = \sum_{P \in V} \nu_P(\omega) P$

Lemma 8.43. *Let ω be a non-zero rational differential on a curve V. Then $\nu_P(\omega) = 0$ for all but finitely many $P \in V$. $\nu_P(\omega) \geq= 0$ iff ω is regular at P.*

Proof. Note that if $\omega, \omega' \in \Omega_V^\times$, then write $\omega = h d\pi_P$, $\omega' = h' d\pi_P$, have $\nu_P(\omega + \omega') = \nu_P(h + h') \geq \min\{\nu_P(h), \nu_P(h')\} = \min\{\nu_P(\omega), \nu_P(\omega')\}$.

First, if $g \in k(V)$ is nonconst and $\omega = f dg$, then for any $P \in V$, have $\nu_P(\omega) = \nu_P(f) + \nu_P(dg)$, so suffices to show $\nu_P(dg) = 0$ for all but finitely many P and any nonconst g. By finiteness for curves, g (when considered as a morphism $V \to \mathbb{P}^1$) has finitely many poles and finitely many ramified pts. If $g(P) \neq \infty$ and t is a local coord at 0 on $\mathbb{P}^1$, then $t - g(P)$ is a local coord at $g(P)$, and f, g is also not ramified at P, and $g^*(t - g(P)) = g - g(P)$ is a local parameter at P, so we have

$$\nu_P(dg) = \nu_P(d(g - g(P))) = \nu_P(1) = 0$$

$\Rightarrow$: clear.

$\Leftarrow$: Suppose $\omega = \sum f_i dg_i$, f_i, g_i ae regular at P. By valuation inequality, suffices to show that $\nu_P(f_i dg_i) \geq 0$ for all i, so suffices to show $\nu_P(dg) \geq 0$ for all $g \in \mathcal{O}_{V,P}^\times$. Have $dg = d(g - g(P)) = d(\pi_P^n h)$, where $n = \nu_P(g - g(P))$ and h is a unit in the local ring. If $n = 1$, then $g - g(P)$ is a local parameter, so $\nu_P(dg) = \nu_P(d(g - g(P))) = 0$. If $n > 1$, then apply Leibniz rule

$$d(\pi_P^n h) = \pi_P^{n-1} d(\pi_P h) + \pi_P h d(\pi_P^{n-1}) = \pi_P^{n-1} d(\pi_P h) + (n-1)\pi_P h \pi_P^{n-2} d\pi_P$$

Each term has valuation $n - 1$, so $\nu_P(d(g - g(P))) \geq n - 1 > 0$. □

So $\operatorname{div}(\omega)$ is a divisor for any $\omega \in \Omega_V^\times$.

Definition 8.44. $K_V = \operatorname{div}(\omega)$ is called a canonical divisor for V.

Lemma 8.45. $\operatorname{div}(\omega) \sim \operatorname{div}(\omega')$ *for $\omega, \omega' \in \Omega_V^\times$.*

Proof. Since Ω_V is a 1-dim, so $\omega = f\omega'$ for some $f \in k(V)^\times$. By defn, $\operatorname{div}(\omega) - \operatorname{div}(\omega') = \operatorname{div}(f)$. □

Definition 8.46. We call the class of K_V in $\operatorname{Cl}(V)$ the canonical class.

Remark 13. Everything in this subsection depends only on the function fields, so isomorphism-invariant.

Example 8.47. $V = \mathbb{P}^1$. Let $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$, $x_1 \mapsto (1 : x_1)$. Define $t = x_1/x_0$, then $\omega = dt$ is a non-zero rational differentia. If $a \in \mathbb{A}^1$, then $t - a$ is a local parameter at a, and $dt = d(t - a)$, $\nu_a(dt) = 0$. At ∞, $1/t = x_0/x_1$ is a local parameter, and we have $d(1/t) = -dt/t^2$, so $\omega = -t^2 d(1/t)$, so $\nu_\infty(\omega) = \nu_\infty(t^2) = -2$, so $\operatorname{div}(\omega) = -2\infty$. Since $\deg(\operatorname{div}(\omega)) < 0$, this is neither principal not a hyperplane section.

Definition 8.48. V a curve. The genus $g(V) = \ell(K_V)$, where K_V is any element of the canonical class.

Note $0 \in L(D)$.

Example 8.49. $g(\mathbb{P}^1) = \ell(-2\infty) = 0$, since degree < 0.

This agrees with the defn in Riemann surfaces if $k = \mathbb{C}$.

8.4 Differentials on Plane Curves

8.4.1 Plane Cubics

Let $F(x_0, x_1, x_2) = x_0 x_2^2 - \prod_{i=1}^3 (x_1 - \lambda_i x_0)$, $\lambda_1, \lambda_2, \lambda_3 \in k$ are distinct. Can check $V = Z(F)$ is irreducible and smooth.

Compute its geneus. Dehomogenize w.r.t. x_0, get $y^2 = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3) = g(x)$, where $y = x_2/x_0$, $x = x_1/x_0$, so $2ydy = g'(x)dx$ in Ω_V. Choose $\omega = dx/y = 2dy/g'(x)$. If $y(P) \neq 0$, then $x - x(P)$ is a loc. coord for $P \in V$ with $x_0 \neq 0$. So $\nu_P(\omega) = \nu_P(dx/y) = 0$. If $y(P) = 0$, thne y is a loc coord at P and λ_i distinct $\implies$ $g'(x(P)) \neq 0$, so $\nu_P(\omega) = 0$. If $x_0 = 0$, have a single point $\infty = (0 : 0 : 1)$. Dehomogenize w.r.t. x_2 to obtain $z - \prod_{i=1}^3 (t - \lambda_i z)$, $z = x_0/x_2$ and $t = x_1/x_2$. t is a local parameter at ∞, i.e., $\nu_\infty(t) = 1$, $\nu_\infty(z) \geq 3$, so $\nu_\infty(t - \lambda_i z) = \nu_\infty(t) = 1$, so $\nu_\infty(z) = 3$. Note that $\nu_\infty(y) = -3$, $\nu_\infty(x) = -2$.

Lemma 8.50. *If $h \in k(V)$ on a curve V, and $\nu_P(h) \geq 2$ for $P \in V$, then $\nu_P(dh) = \nu_P(h) - 1$.*

Proof. If $\nu_P(h) = 1$, then true by defn. If $\nu_P(h) \geq 2$, then write $h = u\pi_P^n$, $n = \nu_P(h)$. Have $dh = u\pi_P d(\pi_P^{n-1}) + \pi_P^{n-1} d(u\pi_P) = u\pi_P^{n-1}(n-1)d\pi_P + \pi_P^{n-1} d(u\pi_P) = \pi_P(n-1)(u(n-1)d\pi_P + ud\pi_P + \pi_P du) = \pi_P^{n-1}[und\pi_P + \pi_P du]$. u is regular at P so $\nu_P(du) \geq 0$, so $\nu_P(\pi_P du) \geq 1$, so $\nu_P(dh) = n - 1$ $(****)$ ☐

$\nu_\infty(1/x) = 2$ so $\nu_\infty(d(1/x)) = 1$, so $\nu_\infty(-dx/x^2) = \nu_\infty(1/x^2) + \nu_\infty(dx) = 1$. Since $\nu_\infty(1/x^2) = 4$, $\nu_\infty(dx) = -3$, so $\nu_\infty(\omega) = \nu_\infty(dx/y) = 0$. So the canonical divisor is the zero divisor, so $L(0) = \{\text{rational functions on } V \text{ with no poles}\} = k$, so $\ell(K_V) = 1 = g(V)$. For a general plane curve, we have

Theorem 8.51. *Let $V = Z(F) \subseteq \mathbb{P}^2$ be a smooth plane curve of degree $d \geq 3$. Then $K_V = (d-3)H$, where H is the divisor of a hyperplane section on V.*

Remark 14. Computation: given $x_0 x_2^2 - \prod_{i=1}^3 (x_1 - \lambda_i x_0)$. The projection to $(x_0 : x_1) \in \mathbb{P}^1$ has deg 2 $[\infty \mapsto (0:1)$ with $\nu_\infty(x) = -2]$; the proj to $(x_0 : x_2)$ has deg 3 $[\infty \mapsto (0:1)$ and $\nu_\infty(y) = -3]$.

Proof. Change coords so $(0 : 1 : 0) \notin V$. Dehomogenize $x = x_1/x_0$ and $y = x_2/x_0$ to obtain $0 = f(x, y) = F(1, x, y)$, so $\partial_x f dx + \partial_y f dy = 0$, so consider $\omega = dx/\partial_y f = -dy/\partial_x f$. Let $H = \{x_0 = 0\}$, so $\text{div}(\omega)$ is supported on H. Now dehomogenizae w.r.t. x_2 to get $g(z, w) = F(z, w, 1)$, $z = x_0/x_2 = 1/y$ and $w = x_1/x_2 = x/y$ and consider $\eta = dz/\partial_w g = -dw/\partial_z g$. $\text{div}(\eta)$ is supported on $\{x_2 = 0\}$ not at any $P \in H \cap V$. Note that $f(x, y) = F(1, x, y) = y^d F(1/y, x/y, 1) = y^d g(1/y, x/y)$. Compute

$$\partial_x f = y^d \partial_w g(1/y, x/y) 1/y = \frac{1}{z^{d-1}} \partial_w g(z, w)$$

Since $y = 1/z$, have $dy = -dz/z^2$, so

$$\omega = \frac{dy}{\partial_x f} = \frac{-z^{-2} dz}{(z)^{1-d} \partial_w g(z, w)} = z^{d-3} \eta$$

Since $\nu_P(\eta) = 0$ for all $P \in H$, we have $\nu_P(\omega) = (d-3)\nu_P(x_0/x_2)$ for all $P \in H$, so $\omega = (d-3)H$ since $H = \text{div}(x_0)$. ☐

So $\deg(K_V) = d(d-3)$ for a smooth plane curve.

Proposition 8.52. *Suppose $f(x, y) = 0$ is an affine equation for a curve $V \subseteq \mathbb{P}^2$. Assume $\deg V \geq 3$. Then $\{x^r y^s : 0 \leq r + s \leq d - 3,\ r, s \geq 0\}$ is a basis for $L(K_V)$*

Proof. By thm, all poles of any elt of $L(K_V)$ are at ∞ (if $x = x_1/x_0$, $y = x_2/x_0$, this means that they satisfy $x_0(P) = 0$). x, y generate $k(V)$ as a field, so any elt of $L(K_V)$ is a polynomial in x, y, so monomials $x^r y^s$ generate $L(K_V)$, since $x^r y^s = \frac{x_1^r x_2^s}{x_0^{r+s}} \in L(K_V)$ iff $0 \leq r + s \leq d - 3$. It suffices show independence. Note that any non-trivial dependence relation would give a degree $< d$ element of the (affine) ideal. Contradiction. ☐

Corollary 8.53 (Genus-degree formula for plane curves)**.** *If V is a smooth plane curve in $\mathbb{P}^2$ of degree ≥ 3. Then $g(V) = \frac{(d-1)(d-2)}{2}$*

Corollary 8.54. *There exists infinitely many non-isomorphic curves.*

9 Riemann-Roch

Theorem 9.1 (Riemann-Roch)**.** *Let V be a curve of genus g with canonical divisor K_V. Then for any divisor D, $\ell(D) - \ell(K_V - D) = 1 - g + \deg(D)$.*

Corollary 9.2 (Genus-degree formula for curves in $\mathbb{P}^2$)**.**

Proof. Take $D - K_V$, so R-R implies $\ell(K_V) - \ell(0) = 1 - g + \deg(K_V)$, so $g - 1 = 1 - g + d(d-3)$. □

(Owen's signature)

Corollary 9.3. *If V is a curve,* $\deg(K_V) = 2g - 2$.

Proof. Take $D = K_V$. Apply Riemann-Roch, so $\ell(D) = g$ and $\ell(K_V - D) - \ell(0) = 1$. □

Corollary 9.4. *Let D be a divisor of* $\deg(D) > 2g - 2$*, then $\ell(D) = 1 - g + \deg(D)$. In particular, if $g = 1$, then* $\deg(D) > 0 \implies \ell(D) = \deg(D)$

Consider $\varphi : V \to W$ a non-constant morphism of curves. $\varphi^* : k(W) \to k(V)$ sends non-constant $t \in k(W)$ to non-constant $\varphi^* t \in k(V)$, so Ω_W is generated by dt and Ω_V by $d\varphi^* t$. Given $\omega = f dt \in \Omega_W$, define $\varphi^*\omega = \varphi^*(f) d\varphi^* t$.

Lemma 9.5. *$\varphi : V \to W$ a non-const morphism of curves. Let $P \in V$, $\varphi(P) = Q$, e_P ramification index of φ at P, π_P, π_Q local parameters at P, Q resp. Then $\nu_P(\varphi^*\omega) = e_P \nu_Q(\omega) + e_P - 1$. In particular, $\nu_P(\varphi^* d\pi_Q) = e_P - 1$.*

Proof. $\nu_P(\varphi^*(d\pi_Q)) = \nu_P(d(\varphi^*\pi_Q)) = e_P - 1$. For general ω, write $\omega = u\pi_Q^{\nu_Q(\omega)} d\pi_Q$, $u \in \mathcal{O}_{W,Q}^\times$. Then $\varphi^*\omega = \varphi^*(u)\varphi^*(\pi_Q)^{\nu_Q(\omega)} d(\varphi^*\pi_Q)$, so $\nu_P(\varphi^*\omega) = e_P\nu_Q(\omega) + e_P - 1$. □

Theorem 9.6 (Riemann-Hurwitz)**.** *Let $\varphi : V \to W$ be a non-constant morphism of curves and let $n = \deg\varphi$. Then $2g(V) - 2 = n(2g(W) - 2) + \sum_{P \in V}(e_P - 1)$.*

Proof. Let $0 \neq \omega \in \Omega_W$, so $\varphi^*\omega \in \Omega_V^\times$, so $\operatorname{div}(\varphi^*\omega)$ is K_V. By Riemann-Roch, $\deg(K_V) = 2g(V) - 2$, so

$$\begin{aligned}
2g(V) - 2 &= \sum_{P \in V} \nu_P(\varphi^*\omega) = \sum_{Q \in W} \sum_{\varphi(P) = Q} \\
&= \sum_{Q \in W} \sum_{\varphi(P) = Q} (e_P \nu_Q(\omega) + e_P - 1) \\
&= \sum_{Q \in W} \deg(\varphi)\nu_Q(\omega) + \sum_{P \in V}(e_P - 1) \\
&= \deg(\varphi)\deg(\operatorname{div}(\omega)) + \sum_{P \in V}(e_P - 1)
\end{aligned}$$

The first term is $n(2g(W) - 2)$. □

Corollary 9.7. *If V, W are curves and $g(V) < g(W)$, then any morphism $\varphi : V \to W$ is constant.*

Corollary 9.8. *Let $X = V_1 \times V_2$ be product of two curves of genus ≥ 1, then X is a smooth projective surface with no subvariety isomorphic to $\mathbb{P}^1$.*

Proof. Riemann-Hurwitz applied to the projection maps. □

9.1 Equations (not embedding???) of Curves

Definition 9.9. Let V be a curve and D a divisor on V with $\ell(D) = n + 1 \geq 2$. Let $B = \{f_0, ..., f_n\}$ be a basis for $L(D)$. The morphism associated to D w.r.t. basis B is $\varphi_D = (f_0 : \ldots : f_n) : V \to \mathbb{P}^n$ (nonconst morphism). We say φ_D is an embedding if it is an isomorphism onto $\varphi_D(V)$ in which case we say D is very ample.

A different choice of basis corresponds to a linear change of coords on $\mathbb{P}^n$, so omit B from notation. Let $V \subseteq \mathbb{P}^n$ be a curve, and $D = \text{div}(x_0)$ a hyperplane section. If $P \neq Q \in V$, then $\ell(D-P-Q) = \ell(D)-2$. The same is true if $P = Q$. Call this $(*)$. D has $(*)$ if $\forall P, Q \in V$, $\ell(D-P-Q) = \ell(D)-2$.

Theorem 9.10. *φ_D is an embedding iff D satisfies $(*)$.*

Proof omitted.

Corollary 9.11. *If $\deg(D) \geq 2g+1$, then φ_D is an embedding.*

Proof. For $D, D-P-Q$, we have $\ell(D) = 1-g-\deg(D)$, $\ell(D-P-Q) = 1-g-\deg(D-P-Q) = 1-g-(\deg(D)-2)$. □

Corollary 9.12. *For fixed genus, if $g \geq 2$, then $\exists m = m(g)$ s.t. mK_V is very ample.*

9.2 Elliptic Curves

Definition 9.13. An elliptic curve is a curve of genus 1 together with a base point $P_0 \in E$ (written $\mathcal{O}_E$).

Suppose $P, Q \in E$ elliptic curve (E, P_0). The divisor $P+Q-P_0$ has degree 1, so $\ell(P+Q-P_0) = 1$, so $\exists$ an effective deg 1 divisor equivalent to $P+Q-P_0$, i.e., $R \in E$ s.t. $P+Q-P_0 \sim R$. Distinct pts on E are inequivalent (since $E \ncong \mathbb{P}^1$. For any curve of genus > 0, if $P \sim Q$, then there exists f with a simple pole, which would imply that f is an isomorphism to $\mathbb{P}^1$), so R is unique. Define $P \oplus_E Q = R$, i.e., $P+Q = R$.

Theorem 9.14. *$\oplus_E$ makes E into an abelian group with identity $\mathcal{O}_E = P_0$. The map $P \mapsto [P-P_0] \in \text{Cl}^0(E)$ is an isomorphism of groups.*

Proof. Define $\beta(P) = [P-P_0]$. If $P, Q \in E$, then $\beta(P+Q) = \beta(R) = [R-P_0] = [P+Q-P_0-P_0] = [P-P_0]+[Q-Q_0] = \beta(P)+\beta(Q)$. β is injective since $P-P_0 \sim Q-P_0 \implies P \sim Q \implies P = Q$. β is surjective [let D be a divisor of deg 0, $D+P_0$ has degree 1, so $\ell(D+P_0) = 1$, so there exists a pt $P \in E$ s.t. $D+P_0 \sim P$, so $D = \beta(P-P_0)$]. □

Remark 15. In genus > 1, the injectivity argument still applies, so V injects into a group. One can show that this group is a projective variety of dimension g, called the Jacobian of V.

(Owen's signature)

Theorem 9.15. *Let (E, P_0) be an elliptic curve, then $3P_0$ provides an embedding of E into $\mathbb{P}^2$ as a plane cubic.*

Proof. If $\deg D = m > 0$, then $\ell(D) = \deg(D)$, so mP_0 is an embedding if $m \geq 3$. $\ell(mP_0) = m$ for $m > 1$, so $L(3P_0)$ has a basis $1, x, y$ where $1, x$ is a basis for $L(2P_0)$, so x is a degree 2 morphism to $\mathbb{P}^1$; y is a degree 3 morphism to $\mathbb{P}^1$ by considering preimages of ∞. So $(1 : x : y) : E \to \mathbb{P}^2$ is an embedding.

1. $L(4P_0)$ has basis $1, x, y, x^2$
2. $L(5P_0)$ has basis $1, x, y, x^2, xy$
3. $L(6P_0)$ contains these and x^3 and y^2, but $\ell(6P_0) = 6$, so there exists a non-trivial linear dependence relation necessarily involving both y^2 and x^3.

Homogenizing, this is a cubic, so $E \subseteq Z(F)$ for a cubic F. The components of $Z(F)$ (if smooth) have genus 0. E is an isomrophism onto its image, so onto a smooth component, which cannot have genus 0, so $Z(F)$ is irreducible and E maps onto $Z(F)$. □

Note P_0 is the unique pt of E sent to line at ∞. We may assume $P_0 = (0:1:0)$ and that the cubic has no yz^2 term. (P_0 unique, $z = 0 \implies F(x,y,0) = x^3$), so $F : ay^2z + byz^2 = G(x,z)$, so can make $b = 0$ by affine change of coord and assume $F : y^2z = \prod_{i=1}^3 (x - \lambda_i z)$.

Theorem 9.16. *Let $E = Z(F)$, $F : y^2z = \prod_{i=1}^3 (x-\lambda_i z)$ an elliptic curve with $P_0 = (0:1:0)$. Then $P+Q+R = \mathcal{O}_E$ iff P, Q, R are collinear.*

Proof. Consider the map β. See that $P+Q+R = \mathcal{O}_E$ iff $P+Q+R \sim 3P_0$ iff $\exists$ rational function f with $\text{div}(f) = P+Q+R-3P_0$. Note $f \in L(3P_0)$. So f is a linear combination of $1, x, y$, i.e., $L(x,y,z)/z$ for a linear L. The zero set of f is precisely P, Q, R, so P, Q, R exists iff such an f exists. □

9.3 More Consequences of Riemann-Roch

Definition 9.17. A curve V of genus $g \geq 2$ is hyperelliptic if $\exists \pi : V \to \mathbb{P}^1$ nonconst of degree 2.

If V is hyperelliptic, consider $D = \pi^*(\infty)$ (poles of π), then $L(D)$ has dim ≥ 2 since $1, \pi \in L(D)$. On the other hand, for $Q \in V$, $\ell(Q) = 1$ since V doesn't have deg 1 map to $\mathbb{P}^1$. So $\ell(D) \leq \ell(D - P) + 1 \leq 2$, so $\ell(D) = 2$.

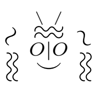
Proposition 9.18. *Let $g(V) > 1$ and suppose there exists an effective divisor D of degree 2 on V with $\ell(D) = 2$. Then $\varphi_D : V \to \mathbb{P}^1$ has degree 2 (and $\varphi_D^*(\infty) = D$), so V is hyperelliptic. All genus 2 curves are hyperelliptic, and there exists hyperelliptic curve of every genus.*

Proof. ES4 □

Proposition 9.19. *$g \geq 3$, φ_{K_V} is an embedding iff V is not hyperelliptic.*

Proof. Suppose φ_{K_V} is not an embedding. Then $\exists P, Q \in V$ s.t. $\ell(K_V - P - Q) \geq g - 1$. $\ell(P + Q) = \ell(K_V - P - Q) + 1 - g + 2 \geq 2$ by R-R. Since $\ell(P + Q) \leq \ell(P) + 1 = 2$. Then the previous proposition implies that V is hyperelliptic.

Conversely, suppose V is hyperelliptic. Then have $\pi : V \to \mathbb{P}^1$ of degree 2 with $\pi^*(\infty)$ effective divisor of degree 2 and $\ell(D) = 2$, so $\ell(K_V - D) \geq g - 1$ working conversely. □



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10 Blow-ups and Completeness of Projective Variety

This section contains the content of a bonus lecture

Definition 10.1. A variety V is complete if for all varieties Y, $\pi_2 : X \times Y \to Y$ is closed.

Theorem 10.2. *Every projective variety is complete. (proper over the base field)*

Blowup of a point on groj space: Let $P = (0 : \cdots : 1) \in \mathbb{P}^n$ and $H = Z(x_n)$. Projection π_P from P. $\pi_P : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ regular away from P.

Definition 10.3. The blowup of $\mathbb{P}^n$ at P is $Bl_P(\mathbb{P}^n) = \overline{\Gamma_{\pi_P}} \subseteq \mathbb{P}^n \times \mathbb{P}^{n-1}$.

Note that $Bl_P(\mathbb{P}^n) = Z(\{x_i y_j - y_i x_j : 0 \leq i, j \leq n - 1\})$. The first projection π_1 is a birational map $Bl_P(\mathbb{P}^n) \to \mathbb{P}^n$ with $\pi^{-1}(Q) = Q$ for all $Q \neq P$, and $\pi^{-1}(P) = \mathbb{P}^{n-1}$. [$Bl_p(\mathbb{P}^n) = \{Q, [L] : Q \in L\} \subseteq \mathbb{P}^n \times \mathbb{P}^{n-1}$. Parametrizes the lines through P.] The second projection $q : Bl_P(\mathbb{P}^n) \to \mathbb{P}^{n-1}$ is the projection of a locally trivial $\mathbb{P}^1$-bundle.

Primary use: birationally identify potentially singular varieties with smooth varieties.

Proof. Step 1: $\mathbb{P}^1$ is complete. First assume $Y = \mathbb{P}^m$. Closed subsets are precisely the zero loci of bihomogeneous polys $F(x_0, x_1, \vec{y}) = a_d(y)x_0^d + a_{d-1}(y)x_0^{d-1}x_1 + \cdots + a_0(y)x_1^d$. Two (one-variable) polys share a root/common factor iff $\mathrm{Res}(f, g) = 0$. Claim that for $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^m = Z(\{\mathrm{Res}(f, g) : f, g$ dehomogenization of $F, G \in I(Z)\})$. Can assume wlog that $Y = \mathbb{P}^m$ and closed subsets in Y are restrictions of closed set in $\mathbb{P}^m$, so $\mathbb{P}^1$ is complete.

It suffices to show that the second proj of $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ is a closed map for all n. Notice that if V is a proj variety and $W \subset V$ a subset with $W \cap U_i$ closed in U_i then W is closed.

$$\begin{array}{ccccc}
\mathbb{P}^1 \times U_i \times \mathbb{P}^m & \longrightarrow & Bl_p(\mathbb{P}^n) \times \mathbb{P}^m & \xrightarrow{\pi \times \mathrm{id}} & \mathbb{P}^n \times \mathbb{P}^m \\
\downarrow & & \downarrow{\scriptstyle q \times \mathrm{id}} & & \downarrow{\scriptstyle \pi_2} \\
U_2 \times \mathbb{P}^m & \longrightarrow & \mathbb{P}^{n-1} \times \mathbb{P}^m & \xrightarrow{\pi_2} & \mathbb{P}^m
\end{array}$$

Assume Z is closed in $\mathbb{P}^n \times \mathbb{P}^m$. $(\pi \times \mathrm{id})^{-1}(Z)$ is closed and has closed preimage in each $\mathbb{P}^1 \times U_i \times \mathbb{P}^m$. Since $\mathbb{P}^1$ is complete, the left arrow is closed. So $(q \times \mathrm{id})(\pi \times \mathrm{id})^{-1}(Z)$ is closed. By inductive hypothesis, $\pi_2 : \mathbb{P}^{n-1} \times \mathbb{P}^m \to \mathbb{P}^m$ is closed, so the second projection of $Z \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is closed in $\mathbb{P}^m$, so $\mathbb{P}^n$ is complete. □