Analysis of Functions

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1 Review of Basic Concepts

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1.1 Probmeas

1. The Lebesgue measure is inner regular, i.e., for all $A \in B(\mathbb{R}^n)$, $\mu(A) = \sup\{\mu(K) : K \subseteq A,\ K \text{ compact}\}$.
2. Recall that μ extends to the μ-completion of $\mathcal{B}$, which equals $M_\mu = \{B \cup A : B \in \mathcal{B}, A \in \mathcal{N},\ \mu(A) = 0\}$.
3. For measurable functions $f : E \to F$, if $(F, \mathcal{F}) = (\mathbb{R}, \mathcal{B})$ (or $(\mathbb{C}, \mathcal{B})$), then we say that f is Borel. This extends to maps taking values $\pm\infty$ if $f^{-1}(\pm\infty) \in \mathcal{E}$. If f takes values in $[0, \infty]$, then we say $f \geq 0$ (non-negative).
4. Recall MCT and DCT.

1.2 L^p-spaces and Approximation

For $f : (E, \mathcal{E}, \mu) \to \mathbb{R}$ (or $\mathbb{C}$), define

$$\|f\|_{L^p} = \left(\int_E |f|^p d\mu\right)^{1/p},\ 1 \leq p < \infty$$
$$\|f\|_{L^\infty} = \operatorname{ess\,sup} |f| = \inf\{\lambda \geq 0 : |f| \leq \lambda \text{ a.e.}\}$$

We use $\|\cdot\|_\infty$ to denote the usual sup-norm. Define $L^p(E, \mu) = \{f : E \to \mathbb{R} : \text{meas. } \|f\|_{L^p} < \infty\}$

Recall Riesz-Fischer Theorem. Also recall the spaces $C^k(\mathbb{R}^n)$, the set of all functions on $\mathbb{R}^n$ with continuous partial derivatives up to order k. We note that $C^\infty(\mathbb{R}^n) = \bigcap_{k\geq 0} C^k(\mathbb{R}^n)$. Note that this includes unbounded smooth functions. Use subscript c to denote the linear subspaces consisting of compactly supported functions.

Remark 1. $C_c^\infty(\mathbb{R}^n)$ is non-empty, e.g.,

$$\psi(x) = \begin{cases} e^{\frac{1}{|x|^2-1}} & |x| < 1 \\ 0 & \text{o/w} \end{cases}$$

Theorem 1.1. *$C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, dx)$ for $1 \leq p < \infty$.*

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We admit the following lemma from PM.

Lemma 1.2. *$C_c(\mathbb{R}^n)$ is dense in L^p, $1 \leq p < \infty$.*

Recall convolution and basic properties including commutativity, associativity, and $\int_{\mathbb{R}^n} f * g dx = \int_{\mathbb{R}^n} f \int_{\mathbb{R}^n} g$ (translation invariance and Fubini).

Recall multi-index notation $\alpha \in \mathbb{Z}_+^n$ is written as $\alpha = (\alpha_1, ..., \alpha_n)$ with order $|\alpha| = \alpha_1 + ... + \alpha_n$ and we set $\alpha! = \alpha_1! \cdots \alpha_n!$, and for $x \in \mathbb{R}^n$, we write $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, so the partial differential operator becomes

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x^\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}$$

In particular $D_i = D^{(0,...,1,0,...,0)}$

Theorem 1.3. *Let $f \in L^1_{loc}$ (i.e., $f 1_K \in L^1$ for any $K \subseteq \mathbb{R}^n$ compact), and $g \in C_c^k(\mathbb{R}^n)$. Then $f * g \in C^k(\mathbb{R}^n)$ and for all $0 \leq |\alpha| \leq k$, we have*

$$D^\alpha(f * g) = f * (D^\alpha g)$$

Proof. Recall the translation operator $\tau_z h = h(\bullet - z)$, $z \in \mathbb{R}^n$. Then for all $u \in \mathbb{R}^n$,

$$\tau_z(f * g)(x) = \int_{\mathbb{R}^n} g(x - u - y) f(y) dy$$

Since $g \in C_c(\mathbb{R}^n)$ we have $|g(x - u - y)| \leq \|g\|_\infty 1_K$ for all $|u| \leq 1$, where $K = K_{x,g}$ is a compact set, so tht $\|g\|_\infty 1_K |f|$ gives an integrable upper bounde for the integrand. Since $g(x - u - y) \to g(x - y)$ as $u \to 0$, we have pointwise convergence. Apply DCT, we see that $f * g$ is cts.

Now for $k = 1$, we define difference operators $\forall e_i$ (standard basis vector) by $\Delta_h^i g(z) = \frac{g(z + he_i) - g(z)}{h}$ which converges to $D_i g(z)$. We can write

$$\Delta_h^i (f * g)(x) = \int_{\mathbb{R}^n} \Delta_h^i g(x - y) f(y) dy$$

Apply mean value inequality, get $|\Delta_h^i g(x - y)| \leq \|D_i g\|_\infty 1_K$. Apply DCT, $\Delta_h^i(f * g) \to f * (D_i g)$, which is continuous, so $f * g \in C^1$. Induction... ☐

Proposition 1.4 (Continuity of translation in L^p)**.** *Let $1 \leq p < \infty$. Then $\|\tau_z f - f\|_{L^p} \to 0$ as $z \to 0$ for all $f \in L^p$.*

Proof. Hold for cts functions with compact support. Then apply $\varepsilon/3$-argument. ☐

Theorem 1.5 (Minkowski's inequality for integrals)**.** *Let $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a measurable non-negative or $dx \otimes dx$-integrable function. Then*

$$\left\| \int_{\mathbb{R}^n} F(x, \cdot) dx \right\|_{L^p} \leq \int_{\mathbb{R}^n} \|F(x, \cdot)\|_{L^p} dx$$

Proof. Example sheet. ☐

Theorem 1.6 (Mollification/Approximate identity)**.** *Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be non-negative s.t. $\int_{\mathbb{R}^n} \varphi(x) dx = 1$. Define $\varphi_\varepsilon^{-n} \varphi(\cdot/\varepsilon)$, $\varepsilon > 0$. Then for $1 \leq p < \infty$ and any $f \in L^p$,*

$$\|f - \varphi_\varepsilon * f\|_{L^p} \xrightarrow{\varepsilon \to 0} 0$$

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Proof. For $f \in L^p$, $x \in \mathbb{R}^n$,

$$\begin{aligned} |\varphi_\varepsilon * f(x) - f(x)| &= \left| \int_{\mathbb{R}^n} f(x - y) \varepsilon^{-n} \varphi(y/\varepsilon) dy - f(x) \right| \\ &= \left| \int_{\mathbb{R}^n} (f(x - \varepsilon u) \varphi(u) - f(x)) du \right| \\ &\leq \int_{\mathbb{R}^n} |f(x - \varepsilon) - f(x)| \varphi(u) du \end{aligned}$$

Apply Minkowski's inequality for integrals,

$$\|\varphi_\varepsilon * f - f\|_{L^p} \leq \int_{\mathbb{R}^n} \|\tau_{\varepsilon u} f = f\|_{L^p} \varphi(u) du$$

This converges to 0 as $\varepsilon \to 0$ by DCT. ☐

In particular, since $C_c(\mathbb{R}^n)$ is dense in L^p and $\{\varphi_\varepsilon * f : f \in C_c(\mathbb{R}^n)\} \subseteq C_c^\infty(\mathbb{R}^n)$, we have also proved that $C_c^\infty(\mathbb{R}^n)$ is dense in L^p.

1.3 Lebesgue's Differentiation Theorem

Definition 1.7 (Hardy-Littlewood maximal function)**.** For $f \in L^1$, $x \in \mathbb{R}^n$, let

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y)dy$$

Lemma 1.8. *For $f \in L^1$, Mf maps $\mathbb{R}^n$ to $\mathbb{R}$ and is Borel-measurable, and for all $\lambda > 0$,*

$$|\{x : Mf(x) > \lambda\}| \leq \frac{3^n}{\lambda}\|f\|_{L^1}$$

Proof. Define $A_\lambda = \{x : Mf(x) > \lambda\}$. If $x_m \in A_\lambda^c$ s.t. $x_m \to x \in \mathbb{R}^n$. Then

$$\frac{1}{|B_{r_x}(x_m)|} \int_{\mathbb{R}^n} 1_{B_{r_x}(x_m)}|f(y)|dy \leq \lambda$$

by definition of A_λ^c. Apply DCT, get a contradiction, so A_λ^c is closed, so A_λ is open. This gives measurability.

To prove the inequality, we use the inner regularity of μ and take an arbitrary compact subset $K \subseteq A_\lambda$. K has an open cover $\{B_{r_x}(x) : x \in A_\lambda\}$. Pass to a finite subcover $B_1, ..., B_N$ of such balls. By Wiener's covering lemma (ES1), reduce to a subcollection osf disjoint balls $B_1, ..., B_k$ s.t.

$$|K| \leq 3^n \sum_{i=1}^k |B_i| = \frac{3^n}{\lambda}\sum_{i=1}^k \lambda|B_i| \leq \frac{3^n}{\lambda}\sum_{i=1}^k \int_{B_i} |f(y)|dy \leq \frac{3^n}{\lambda}\|f\|_{L^1}$$

By inner regularity, $|A_\lambda| \leq \sup\{|K| : K \subseteq A_\lambda \text{ cpt}\} \leq \frac{3^n}{\lambda}\|f\|_{L^1}$. □

Theorem 1.9. *Let $f \in L^1(\mathbb{R}^n)$, $B_r(x)$ ball centered at x with radius r. Then*

$$\lim_{r\to 0} \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y) - f(x)|dy = 0, \text{ a.e.} \tag{†}$$

Remark 2. The set of points $A = \{x \in \mathbb{R}^n : (†)\}$ are called Lebesgue points of f.

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Proof. Consider

$$\bar{A}_\lambda = \left\{x : \lim_{r\to 0} |B_r(x)^{-1}| \int_{B_r(x)} |f(y) - f(x)|dy > 2\lambda\right\}$$

Let $\varepsilon > 0$. Pick $g \in C_c(\mathbb{R}^n)$ s.t. $\|f - g\|_{L^1} < \varepsilon$. Then

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y)-f(x)|dy \leq \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y)-g(y)|dy + \frac{1}{|B_r(x)|}\int_{B_r(x)} |g(y)-g(x)|dy + |f(x)-g(x)|$$

g is unif. cts, so the second term is small. If $x \in \bar{A}_\lambda$, either the first term or the third term is $> \lambda$. The third term is bounded using Markov's inequality

$$\{x : |f(x) - g(x)| > \lambda\} \leq \frac{\|f - g\|_{L^1}}{\lambda} < \varepsilon/\lambda$$

For the first term, use HL-maximal inequality,

$$|\{x : \text{first term} > \lambda\}| \leq |\{x : M(f-g)(x) > \lambda\}| \leq \frac{3^n}{\lambda}\|f - g\|_{L^1} \leq 3^n\varepsilon/\lambda$$

Therefore $|\bar{A}_\lambda| \leq C\varepsilon$. So $|A^c| \leq |\bigcup_n \bar{A}_{1/n}| \leq \sum_n |\bar{A}_{1/n}| = 0$. □

Remark 3. In particular, for $f \in L^1(\mathbb{R})$, $\lim_{h\to 0} \int_x^{x+h} f(y)dy = f(x)$ a.e.

Theorem 1.10 (Egorov)**.** *Let $E \in B(\mathbb{R}^n)$, $|E| < \infty$. Suppose $f_j : E \to \mathbb{R}$ measurable s.t. $f_j \to f$ a.e. on E. Then*

$$\forall \varepsilon > 0,\ \exists A_\varepsilon \text{ s.t. } |E \setminus A_\varepsilon| < \varepsilon \text{ and } f_j \overset{unif}{\to} f \text{ on } A_\varepsilon$$

Proof. By discarding a null set, we may assume that $f_j \to f$ pointwise on E. Define

$$E_k^m = \{x : \forall j > k,\ |f_j(x) - f(x)| < 1/m\}$$

E_k^m is increasing as $k \to \infty$, and $\bigcup_k E_k^m = E$ by pointwise convergence. Pick a subsequence k_m s.t. $|E \setminus E_{k_m}^m| \leq \varepsilon 2^{-m}$. Define $A_\varepsilon = \bigcap_m E_{k_m}^m$. For all $x \in A_\varepsilon$, $|f_j(x) - f(x)| < 1/m$ whenever $j > k_m$, so the convergence is uniform on A_ε, and

$$|E \setminus A_\varepsilon| \leq \sum_m |E \setminus E_{k_m}^m| \leq \varepsilon$$

□

Theorem 1.11 (Lusin)**.** *Let $|E| < \infty$, $f : E \to \mathbb{R}$ (or $\mathbb{C}$) Borel-measurable. Then*

$$\forall \varepsilon > 0,\ \exists F_\varepsilon \text{ s.t. } |E \setminus F_\varepsilon| < \varepsilon \text{ and } f : F_\varepsilon \to \mathbb{R} \text{ cts}$$

Remark 4. Note that f is not necessarily continuous F_ε when regarded as a map defined on E.

Proof. First prove it for simple functions $f = \sum_{i=1}^m a_i 1_{A_i}$ (wlog assume A_i disjoint), where $\bigcup A_i = E$. Use inner regularity to find compact sets $K_k \subseteq A_k$ s.t. $|A_k \setminus K_k| < \varepsilon/m$. f is cts on $\bigcup_k K_k$ and $|E \setminus \bigcup K_k| \leq \varepsilon$. For general f, approximate f ptwise by simple functions on E. Pick A_ε s.t. $|E \setminus A_\varepsilon| < \varepsilon/2$ s.t. $f_m \to f$ unif. by Egorov. Take C_m compact s.t. $|E \setminus C_m| < \varepsilon 2^{-m-1}$. Then Take $F_\varepsilon = A_\varepsilon \cap \bigcap_m C_m$. Can check that $|E \setminus F_\varepsilon| \leq \varepsilon$. □

Recall Riesz representation theorem in Hilbert spaces (bounded linear functionals can be written as taking inner product with a certain element).

Consider two measures μ, ν on a measurable space $(E, \mathcal{E})$.

Definition 1.12. We say that ν is absolutely continuous w.r.t μ if $\mu(A) = 0 \implies \nu(A) = 0$ for any $A \in \mathcal{E}$. We write $\nu \ll \mu$. If $\nu \ll \mu$ and $\mu \ll \nu$ both hold, then we say that μ, ν are mutually absolutely continuous.

If there exists $B \in \mathcal{E}$ s.t. $0 = \mu(B) = \nu(B^c)$, then we say that μ, ν are mutually singular, and we write $\mu \perp \nu$.

Theorem 1.13 (Radon-Nikodym)**.** *Let μ, ν be finite measures on $(E, \mathcal{E})$ s.t. $\nu \ll \mu$. Then $\exists w \in L^1(\mu)$, $w \geq 0$, s.t. for all $A \in \mathcal{E}$, $\nu(A) = \int_A d\nu = \int_A w d\mu$.*

Remark 5.

1) w is unique.

2) Will show that $\int_E h d\nu = \int_E h w d\mu$ for all $h \geq 0$ measurable. In particular, $w = d\nu/d\mu$ (Leibniz notation) is called the Radon-Nikodym derivative (or density) of ν w.r.t. μ.

3) The result extends to μ, ν σ-finite.

Proof (von Neumann). Define $\alpha = \mu + 2\nu$ and $\beta = 2\mu + \nu$. On $L^2(\alpha)$, consider the map $\Lambda(f) = \int_E f d\beta$. This is bounded since

$$|\Lambda(f)| \leq \int_E |f| d\beta \leq 2 \int_E |f| d\alpha \leq 2\sqrt{\alpha(E)} \|f\|_{L^2(\alpha)}$$

By Riesz, there exists $g \in L^2(\alpha)$ s.t. $\Lambda(f) = \int_E g f d\alpha$, i.e., $\int f(2d\mu + d\nu) = \int g f(d\mu + 2d\nu)$ for all $f \in L^2(\alpha)$. Rearrange,

$$\int_E f(2 - g) d\mu = \int_E f(2g - 1) d\nu$$

Consider $A_j = \{x : g(x) \leq \frac{1}{2} - \frac{1}{j}\}$, $j \in \mathbb{N}$. Thus, by taking $f = 1_{A_j}$,

$$\frac{3}{2}\mu(A_j) \leq \int_E f(2g - 1) d\mu \leq -\frac{2}{j}\nu(A_j)$$

So, $g \geq 1/2$ a.e. (w.r.t. both μ, ν). Similarly, by considering $\{x : g(x) \geq 2 + 1/j\}$, can prove that $g \leq 2$ μ-, ν- a.e. We extend to simple functions and then to non-negative measurable functions by MCT.

Consider $f = 1_{\{x:g(x)=1/2\}}$, then get $\frac{3}{2}\mu(\{x : g(x) = 1/2\}) = 0$, so $\nu(\{x : g(x) = 1/2\}) = 0$.

Let $h \geq 0$ measurable and define $f = \frac{h}{2g-1}$ and $w = \frac{2-g}{2g-1}$ (define it to be 0 if $2g - 1 = 0$). Now

$$\int_E h d\nu = \int_E f(2g-1) d\nu = \int_E f(2-g) d\mu = \int_E hw d\mu$$

Done by taking $h = 1_A$. Note that $w \in L^1(\mu)$ since $\int_E w d\mu = \int_E 1_E w d\mu = \nu(E) < \infty$. □

Remark 6.

1) If $\mathbb{P}$ is a prob measure on B s.t. $\mathbb{P} \ll dx$, where dx is the Lebesgue measure, then $\frac{d\mathbb{P}}{dx} = p(x)$ is the Lebesuge prob density of $\mathbb{P}$. Moreover, there exists a unique decomposition $\mathbb{P} = \mathbb{P}_{\ll} + \mathbb{P}_{\perp}$. (ESheet)

2 Dual Spaces

Definition 2.1. Let X be a topological vector space. The top dual space is

$$X' = \{\Lambda : X \to \mathbb{R} \text{ linear and cts}\}$$

If $(X, \|\cdot\|)$ is a normed space, then cts$\Leftrightarrow$bounded, and X' has the operator norm. $X'' = (X')'$ is called the bidual space.

We have a point evaluation map $\Lambda \mapsto \Lambda(x)$ for each $x \in X$. [Note that $|\Lambda(x)| \leq \|\Lambda\|_{X'}\|x\|_X \leq C\|\lambda\|_{X'}$.] Can identify $x \mapsto (\Lambda \mapsto \Lambda(x))$ and regard X as a subspace of X''.

Definition 2.2. If $X'' = X$, then say X is reflexive.

For $1 < p < \infty$, consider the Hölder conjugate q of p. Each $g \in L^q$ defines a linear functional on L^p by $\Lambda_g(f) = \int fg dx$. This is bounded since $|\Lambda_g(f)| \leq \|g\|_{L^q}\|f\|_{L^p}$. In fact $\|\Lambda_g\| = \|g\|_{L^q}$. Get an embedding $L^q \subseteq (L^p)'$.

Theorem 2.3. *For $1 \leq p < \infty$, $(L^p)' = L^q$. For $1 < p < \infty$, L^p is reflexive.*

Remark 7. The result is false for $p = \infty$, so L^1 is not reflexive.

Lemma 2.4. *Under the hypothesis of the theorem, let $U \in (L^p)'$ positive, then $\exists g \in L^q$ s.t. $U(f) = \Lambda_g$ and $\|U\| = \|g\|_{L^q}$.*

Proof of lemma. On $\mathbb{R}^n$ consider the finite measure μ with density $e^{-|x|^2}$. Further define, for each $A \in \mathcal{B}$, the set function

$$\nu(A) = U(e^{-|x|^2/p} 1_A) \geq 0$$

To show that ν is countably additive, consider $A_m \in \mathcal{B}$ s.t. $A_m \downarrow \bigcap_m A_m = \varnothing$ and note $\nu(A_m) = U(e^{-|x|^2/p}1_{A_m}) \leq \|U\|\|\|e^{-|x|^2/p}1_{A_m}\|_{L^p} \to 0$ by DCT. Hence ν is a finite measure. Note that $\nu \ll \mu$. [If $\mu(A) = 0$, then $\nu(A) \leq \|U\|\|\|e^{-|x|^2/p}\|_{L^p} = \|U\|\mu(A)^{1/p} = 0$.] By Radon-Nikodym, $\exists \mathcal{G} \in L^1(\mu)$ non-negative s.t. $\nu(A) = \int_A \mathcal{G} d\mu$. Consider a simple function $F = \sum_k a_k 1_{A_k}$. Compute

$$U(e^{-|x|^2/p}F) = \int \sum_k a_k 1_{A_k} \mathcal{G} e^{-|x|^2} dx = \int e^{-|x|^2/p} \sum_k a_k 1_{A_k} \mathcal{G} e^{-|x|^2/q} dx$$

Set $g = \mathcal{G}e^{-|x|^2/q}$. Note that $\{e^{-|x|^2/p}F : F \text{ simple}\}$ is dense in $L^p \cap \{\geq 0\}$. Since $fg \in L^1$ [Note $\int |fg| = \int |f|g = U(|f|) \leq \|U\|\|f\|_{L^p} < \infty$]. Decomposing fg into $f_+g - f_-g$ and taking limits, we see that $U(f) = \int fg$ for all $f \in L^p$.

(cf. ES1), have $\|g\|_{L^q} = \sup\{\int |fg| : \|f\|_{L^p} \leq 1\} = U(|f|) \leq \|U\| < \infty$ and $\|U\| = \sup_{\|f\|_{L^p} \leq 1} |\int fg| \leq \|g\|_{L^q}$ by Hölder, so $\|U\| = \|g\|_{L^q}$. □

Proof of thm. Note (ES) that $\Lambda \in (L^p)'$ can be uniquely decomposed as $\Lambda_+ - \Lambda_-$, where $\Lambda_\pm$ are positive linear functionals ($\Lambda_\pm(h) \geq 0$ for all $h \geq 0$ a.e.). Apply the preceding lemma. □

We can characterize duals of subspace of L^∞, e.g., any finite measure defines a linear functional in $C_c(\mathbb{R}^n)'$ by $f \mapsto \mu(f)$.

Definition 2.5. A measure is regular on $\mathbb{R}^n$ if $\forall \varepsilon > 0, \forall A \in \mathcal{B}, \exists C$ closed, D open s.t. $C \subseteq A \subseteq D$ s.t. $\mu(D \setminus C) < \varepsilon$

Theorem 2.6 (Riesz)**.** *Let $\Lambda \in (C_c(\mathbb{R}^n))'$ be positive. Then $\exists$ a σ-algebra $M \supseteq \mathcal{B}$ and a regular measure μ on M s.t. $\Lambda(f) = \int_{\mathbb{R}^n} f d\mu$.*

Proof omitted.

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3 Weak and Weak∗ Topology

Definition 3.1. A semi-norm p on a vector space X is a functional $p : X \to [0, \infty)$ s.t.

1) $\forall x, y \in X,\ p(x + y) \leq p(x) + p(y)$;
2) $\forall x \in X, \forall \lambda \in \mathbb{R}$ (or $\mathbb{C}$), $p(\lambda x) = |\lambda| p(x)$

The collection $\mathcal{P}$ of seminorms introduces a 'locally convex' topology $\tau_{\mathcal{P}}$ generated by

$$V_x(p, n) = \{y \in X : p(y - x) < 1/n\}$$

for $x \in X$, $p \in \mathcal{P}$, $n \in \mathbb{N}$.

Definition 3.2. The family $\mathcal{P}$ is said to separate points if for any $0 \neq x \in X$, there exists $p \in \mathcal{P}$ s.t. $p(x) \neq 0$.

Therefore (ES) a sequence x_n converges in $\tau_{\mathcal{P}}$ iff for all $p \in \mathcal{P}$, $p(x_n - x) \to 0$ as $n \to \infty$. This topology is not generally metrizable unless $\mathcal{P}$ is countable. In that case a metric is given by

$$d_{\mathcal{P}}(x, y) = \sum_{i=1}^{\infty} \frac{p_i(x - y)}{2^{-i}(1 + p_i(x - y))}$$

Definition 3.3. We say that $(X, \tau_{\mathcal{P}})$ is a locally convex topological vector space (LCTVS). If it's complete, then we call it a Frechet space.

Consider the semi-norms given by $p_\Lambda(x) = |\Lambda(x)|$.

Definition 3.4. The topology $\tau_{\mathcal{P}}$ induced by $\mathcal{P} = \{p_\Lambda : \Lambda \in X'\}$ is called the weak topology τ_w. We say that $x_n \to x$ weakly in X or $x_n \rightharpoonup x$ if $\Lambda(x_n) \to \Lambda(x)$ for all $\Lambda \in X'$.

Definition 3.5. On the dual space X', we can consider the weak-∗ topology τ_{w^*} induced by $\mathcal{P} = \{p_x(\Lambda) = |\Lambda(x)| : x \in X\}$. Note that $\Lambda_n \to \Lambda$ weak-∗, or $\Lambda_n \rightharpoonup^* \Lambda$ if $\Lambda_n(x) \to \Lambda(x)$ for all $x \in X$.

Example 3.6.

1) Consider $L^p(\mathbb{R}, dx)$. $f_n \to f$ weakly in L^p iff

$$\forall g \in L^q,\ \int_{\mathbb{R}^n} f_n g dx \to \int_{\mathbb{R}^n} f g dx \tag{†}$$

Since $L^p = (L^q)'$, $\Lambda_{f_n} \to \Lambda_f$ weak-∗ (or $f_n \to f$ weak-∗) iff (†) holds. For $1 < p < \infty$, weak convergence and weak-∗ convergence coincide. (This is true in any reflexive space.)

2) Consider a probmeas on a metric space D (with Borel σ-algebra). Let $C_b(D)$ denote the Banach space of bounded cts functions on D. Then $\mu(f)$ defines an element $C_b(D)'$. A sequence of probmeas μ_n converges to μ in τ_{w^*} if $\mu_n(f) \to \mu(f)$ for all $f \in C_b(D)$. (i.e. weak convergence of laws)

Recall Arzela-Ascoli. A sufficient condition for equicontinuity is given by Hölder continuity, defined as

$$\|f\|_{C^{0,\gamma}} = \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\gamma}$$

where $0 < \gamma < 1$ and

$$\|f\|_{C^{m,\gamma}} = \sum_{0 \leq |\alpha| \leq m} \|D^\alpha f\|_\infty + \max_{|\alpha| = m} \sup_{x \neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x - y|^\gamma}$$

So $\{f : \|f\|_{C^{0,\gamma}} \leq 1\}$ is compact in $C([0, 1])$ by Arzela-Ascoli.

Theorem 3.7 (Banach-Alaoglu). *Let X be a normed space. The unit ball $B_1 = \{\Lambda \in X' : \|\Lambda\|_{X'} \le 1\}$ of X' is compact in weak-$*$ topology*

Remark 8. In $(C_b(D))'$ any sequence of probmeas has a weak-$*$ convergent subsequence.

We will prove Banach Alaoglu for X separable.

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Lemma 3.8. *For a countable dense subset $D = \{x_1, ..., x_n\}$ of X, consider seminorms $\tilde{\mathcal{P}} = \{p_{x_k}(\Lambda) = |\Lambda(x_k)| : k \in \mathbb{N}\}$ with induced topology $\tau_{\tilde{\mathcal{P}}}$. Then $\tau_{w^*} = \tau_{\tilde{\mathcal{P}}}$ coincide as topologies on $B_1' = \{\Lambda \in X' : \|\Lambda\|_{X'} \le 1\}$ and are metrized by*

$$d_{\tilde{\mathcal{P}}(\Lambda,\Lambda')} = \sum_{k=1}^{\infty} \frac{|\Lambda(x_k) - \Lambda'(x_k)|}{2^k(1 + |\Lambda(x_k) + \Lambda'(x_k)|)}$$

Proof. The open sets for $\tau_{\tilde{\mathcal{P}}}$ are generated by $V(x_k, m) = \{\Lambda : |\Lambda(x_k)| < 1/m\}$. To prove that the two topologies are equivalent, it suffices to show that $V(x, n)$ contain some $V(x_k, m)$ for all $x \in X, n \in \mathbb{N}$. Suppose $x \in X \setminus D$ and pick $x_k \in D$ s.t. $\|x - x_k\| < \varepsilon$. For $\Lambda \in V(x_k, m)$ we have $|\Lambda(x)| \le |\Lambda(x_k - x)| + |\Lambda(x_k)| \le \|\Lambda\|_{X'}\varepsilon + 1/m < n$ when ε is sufficiently small and m sufficiently large, so $V(x_k, m) \subseteq V(x, n)$.

If $\Lambda_j(x_k) \overset{j\to\infty}{\longrightarrow} \Lambda(x_k)$ for all k then $d_{\tilde{\mathcal{P}}}(\Lambda_j, \Lambda) \to 0$ by DCT applied to counting measure on $\mathbb{N}$. □

Theorem 3.9. *Let $\Lambda_j \in B_1'$. Then $\exists \Lambda \in B_1'$ s.t. $\Lambda_{j_k} \to \Lambda$ weak-$*$.*

Proof. Let D be a countable dense subset of X. Since $|\Lambda_j(x_k)| \le \|x_k\| < \infty$. Diagonalization argument. Find convergent subsequences $\Lambda_{i,j}$ Find a Λ as the limit of $\Lambda_{j,j}$. Need to show linearity and continuity. Note that Λ is unif cts on D. [If $x, y \in D$ with $\|x - y\| < \varepsilon/2$, then for all j sufficiently large, $|\Lambda_{j,j}(x) - \Lambda(x)|, |\Lambda(y) - \Lambda_{j,j}(y)| < \varepsilon/4$] Apply triangle inequality to $|\Lambda(x) - \Lambda(y)|$. Note that $\Lambda_{j,j}$ is uniformly Lipschitz. By uniform continuity, we can extend Λ to a unif cts function on X.

To show linearity, let $x, y \in X$, $z = x + ay$ for $a \in \mathbb{R}$ (or $\mathbb{C}$) and pick $x', y', z' \in D$ s.t. $\|x - x'\| + |a|\|y - y;\| + \|z - z'\| < \delta$.

Apply a big triangle inequality.

$$\begin{aligned} |\Lambda(z) - \Lambda(x) - a\Lambda(y)| &\le |\Lambda(z) - \Lambda(z')| + |\Lambda(x) - \Lambda(x')| + |a||\Lambda(y) - \Lambda(y')| \\ &\le +|\Lambda(z') - \Lambda_{j,j}(z')| + \text{ ....}| \end{aligned}$$

each term is small either by unif continuity or unif convergence or linearity of Λ on D.

Need to show that $\|\Lambda\| \le 1$ ($|\Lambda(x)| \le |\Lambda(x - x')| + |\Lambda(x')|$) and that the convergence holds on X. □

4 The Hahn-Banach theorem and its consequences

Definition 4.1. A functional $p : X \to \mathbb{R}$ on a real vector space is called sub-linear if

(i) $p(x + y) \le p(x) + p(y)$ for all $x, y \in X$

(ii) $p(tx) = tp(x)$ for all $t \ge 0$, $x \in X$.

Lemma 4.2 (Bounded extension). *Let X be a real vector space and $p : X \to \mathbb{R}$ sub-linear Let $M \subsetneq X$ be a vector subspace, and for $x \in X \setminus M$ define $\tilde{M} = \operatorname{span}(M, x) = \{M + cx : c \in \mathbb{R}\}$. If $l : M \to \mathbb{R}$ is a linear form s.t. $l(x) \le p(x)$ for all $x \in M$, then there exists $\tilde{l} : \tilde{M} \to \mathbb{R}$ linear s.t. $\tilde{l}_M = l$ and $\tilde{l}(x) \le p(x)$ for all $x \in \tilde{M}$.*

Proof. Let $y_1, y_2 \in M$. $l(y_1) + l(y_2) = l(y_1 + y_2) \le p(y_1 + y_2) \le p(y_1 - x) + p(y_2 + x)$ for all $x \in X \setminus M$. Rearrange, $l(y_1) - p(y_1 - x) \le l(y_2) - p(y_2 + x)$. Take sup/inf,

$$\sup\{l(y) - p(y - x) : y \in M\} \le a \le \inf\{p(y + x) - l(y) : y \in M\} \tag{$*$}$$

for some $a \in \mathbb{R}$. If $z \in \tilde{M}$, then it has a unique decomposition $z = y + \lambda x$ for some $\lambda \in \mathbb{R}$. Define $\tilde{l}(z) = \tilde{l}(y + \lambda x) = l(y) + \lambda a$. To see $\tilde{l} \le p$ on $\tilde{M}$, for $\lambda > 0$, write $\tilde{l}(y + \lambda x) = \lambda(l(y/\lambda) + a) \overset{(*)}{\le} \lambda(l(\frac{y}{\lambda}) + p(\frac{y}{\lambda} + x) - l(\frac{y}{\lambda})) = p(y + \lambda x)$. For $\lambda < 0$, let $\mu = -\lambda$ and $\tilde{l}(y + \lambda x) = \mu(l(\frac{y}{\mu} - a)) \le \mu(l(\frac{y}{\mu}) - l(\frac{y}{\mu}) + p(\frac{y}{\mu} - x))$. □

To extend l to all of X (for X separable), we can apply the extension lemma inductively to $M_n = \text{span}(M; x_1, ..., x_n)$, where $(x_n)_{n\in\mathbb{N}}$ is a countable dense subset of X.

In general, consider $S = \{(N, \tilde{l}) : M \subseteq N \subseteq X \text{ vec.sp. }, \tilde{l}|_M = l, \ \tilde{l} \leq p \text{ on } N\}$. Apply Zorn's lemma.

Theorem 4.3 (Hahn-Banach)**.** *Let X be a real vector space and $p : X \to \mathbb{R}$ a sublinear functional. For $M \subseteq X$ vec. subspace, let $l : M \to \mathbb{R}$ be a linear functional s.t. $l(x) \leq p(x)$ for all $x \in M$. Then there exists an extension $\tilde{l} : X \to \mathbb{R}$ (linear) s.t. $\tilde{l}(x) \leq p(x)$ for all $x \in X$.*

Remark 9. Extensions need not be unique. If X is non-separable, the result depends on Axiom of Choice.

Corollary 4.4 (Norming Functional)**.** *Let X be a normed linear space. For all $x \in X$, there exists a linear functional $\Lambda = \Lambda_x \in X'$ s.t. $\|\Lambda\| = 1$ and $|\Lambda(x)| = \|x\|_X$. In particular, if $\Lambda(x - y) = 0$ for all $\Lambda \in X'$, then $x = y$.*

$\subset]- : \langle$ (Owen's (infinitely handsome) signature)

Proof. For $x \in X$ define the vector subspace $M = \{cx : c \in \mathbb{R}\}$ and consider the linear functional $l(cx) = c\|x\|_X$, so $|l(y)| \leq p(y) = \|y\|_X$. By Hahn-Banach, there exists $\Lambda = \Lambda_x : X \to \mathbb{R}$ s.t. $|\Lambda_x(y)| \leq \|y\|_X$, so $\Lambda \in X'$ and $\|\Lambda\| \leq 1$. Note that $\|\Lambda\|_{X'} \geq \sup_{y\in M\cap B_X} |l(y)| \geq \|x\|_X$ □

Corollary 4.5. *The canonical injection of $i : X \hookrightarrow X''$ given by $x \mapsto (\Lambda \to \Lambda(x))$ is an isometric embedding.*

Proof. Consider $\|i(x)\|_{X''} = \sup_{\|\Lambda\|\leq 1} |\Lambda(x)| \leq \|x\|_X$. By taking a norming functional, $\|i(x)\|_{X''} \geq \|\Lambda_x(x)\| = \|x\|$. □

If X is reflexive, then X is isometrically isomorphic to $(X')'$ which is complete. (i.e., reflexive normed lienar space is Banach.) If X is not reflexive, then X'' provides (up to iso) the completion of X for $\|\cdot\|_X$.

In particular, if X is reflexive ,then the weak topology coincides with the weak-$*$ topology on $(X')'$, so Banach-Alaoglu the unit ball B_X is compact in τ_w.

Theorem 4.6 (Hyperplane Separation)**.** *Let A, B be non-empty disjoint convex sets in a Banach space X over $\mathbb{R}$.*

(i) If A is open, then $\exists \Lambda \in X'$ and $\gamma \in \mathbb{R}$ s.t. $\Lambda(a) < \gamma \leq \Lambda(b)$ for all $a \in A$ and $b \in B$.

(ii) If A is compact and B is closed, then $\exists \Lambda \in X'$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ s.t. $\Lambda(a) < \gamma_1 < \gamma_2 < \Lambda(b)$ for all $a \in A$, $b \in B$.

Proof. (i): Pick $a_0 \in A$, $b_0 \in B$ and let $x_0 = b_0 - a_0$. Define $C = A - B + x_0$. C is convex, $0 \in C$, $x_0 \notin C$. C is open. Consider the Minkowski functional defined as

$$p_C(x) = \inf\{t > 0 : x/t \in C\}$$

Can show (ES) that p_C is

- sublinear on X
- there exists $k > 0$ s.t. $p_C(x) \leq k\|x\|$
- $p_C(x) < 1$ for $x \in C$ and $p_C(x) \geq 1$ for $x \notin C$.

Take

$$M = \{tx_0 : t \in \mathbb{R}\}$$

and consider the linear functional $l : M \to \mathbb{R}, \ tx_0 = t$. Then l is dominated by p_C since

$$l(tx_0) = t \leq tp_C(x_0) = p(tx_0) = p_C(tx_0)$$

for $t > 0$ and $l(tx_0) = t \leq 0 \leq p_C(tx_0)$. By Hahn-Banach, there exists $\Lambda : X \to \mathbb{R}$ s.t. $-k\|x\| \leq -p_C(x) \leq \Lambda(x) \leq p_C(x) \leq k\|x\|$ for all $x \in X$, so $\Lambda \in X'$. Pick $a \in A$, $b \in B$. Note that

$$\Lambda(a) - \Lambda(b) + \Lambda(x_0) = \Lambda(a - b + x_0) \leq p_C(a - b + x_0) < 1$$

So $\Lambda(a) < \sup \Lambda(A) \leq \Lambda(b)$.

(2): $\Lambda(A)$ is compact in $\mathbb{R}$ and $d = \|A - B\|_X$. Then consider $\tilde{A}_d = A + B_{d/2}$, where $B_{d/2} = \{y : \|y\| < d/2$ still disjoint from B. Apply (1). □

??? (Owen's missing Signature)

5 Generalized Functions and Distributions

Consider a topological vec. space $X \subseteq \bigcap_{q\geq 1} L^q(\Omega, dx)$, where Ω is an open subset of $\mathbb{R}^n$. Suppose X contains $C_c^\infty(\Omega)$. Let $f \in L^q$, then obtain a linear functional on X given by $\Lambda_f(g) = \int_\Omega fg dx$, $g \in X$. If the embedding $X \hookrightarrow L^q$ is cts, then $\Lambda_f \in X'$. Note that $g = \phi_\epsilon$ is contained in X, mollification implies $\Lambda_f = 0 \implies f = 0$ a.e.. So we can identify Λ_f with f and study the weak-$*$ topology of X' on L^p.

Define seminorms on $C^\infty(\Omega)$, $p_N(\phi) = \max_{0\leq|\alpha|\leq n} \sup_{x\in K_N} |D^\alpha \phi(x)|$, where $K_i \subseteq K_{i+1}$ and $\bigcup_i K_i = \Omega$. We define the Frechet space $\mathcal{E}(\Omega) = (C^\infty(\Omega), \tau_\mathcal{P})$, where $\mathcal{P} = \{p_N : N \in \mathbb{N}\}$. [Note that $\mathcal{E}(\Omega)$ may contain non-integrable functions.]

Theorem 5.1. *There exists a topology τ on $C_c^\infty(\Omega)$ s.t.*

(1) vector space operations are cts

(2) a sequence $\phi_j \overset{j\to\infty}{\to} 0$ iff $\exists K \subseteq \Omega$ compact s.t. $\operatorname{supp}(\phi_j) \subseteq K$ for all j and $D^\alpha \phi_j \to 0$ unif. on K for all $0 \leq |\alpha| < \infty$.

(3) If $T : C_c^\infty(\Omega) \to \mathbb{R}$ (or $\mathbb{C}$) is linear, then it's cts iff $T(\phi_j) \to 0$ for all $\phi_j \to 0$ in τ.

proof omitted.

Definition 5.2. We define $\mathcal{D} = \mathcal{D}(\Omega) = (C_c^\infty(\Omega), \tau)$, the space of test functions.

For each $\phi \in C_c^\infty(\Omega)$, define $e^{-j}\phi(\cdot/j)$, then $e^{-j}\phi(\cdot/j) \to 0$ in D, but $j^{-2025}\phi(\cdot/j)$ does not converge to 0 in D.

Definition 5.3. Call $\phi \in C^\infty(\mathbb{R}^n)$ rapidly decreasing if $\sup_{x\in\mathbb{R}^n} (1 + |x|)^N |D^\alpha \phi(x)| < \infty$ for all $0 \leq |\alpha| < \infty$ and all $N \in \mathbb{N}$.

[Note that $e^{-|x|^2}$ is rapidly decreasing but $(1 + |x|)^{-2025}$ is not.]
Define seminorms $\tilde{\mathcal{P}} = \{\tilde{p}_N : N \in \mathbb{N}\}$ with

$$\tilde{p}_N = \max_{0\leq|\alpha|\leq N} \sup_{x\in\mathbb{R}^n} (1 + |x|)^N |D^\alpha \phi(x)|$$

Define Frechet space $\mathcal{S}(\mathbb{R}^n) = (\{\phi \text{ rapidly decreasing}\}, \tau_{\tilde{\mathcal{P}}})$. This is metrizable since $\tilde{\mathcal{P}}$ is countable. This is called the Schwartz class.

Clearly $\mathcal{D}(\Omega) \subsetneq \mathcal{E}(\Omega)$, $D(\mathbb{R}^n) \subsetneq \mathcal{S}(\mathbb{R}^n) \subsetneq \mathcal{E}(\mathbb{R}^n)$ with continuous embedding (ES). We can now define $\mathcal{D}'(\Omega) = \{T : \mathcal{D}(\Omega) \to \mathbb{R} \text{ (or } \mathbb{C}) \text{ linear and cts}\}$, the space of Schwartz distributions. We also define $\mathcal{E}(\Omega) = \{T : \mathcal{E}(\Omega) \to \mathbb{R} \text{ (or } \mathbb{C}) \text{ linear and cts}\}$ the space of compactly supported Schwartz distributions. Finally for $\Omega = \mathbb{R}^n$ we define $\mathcal{S}(\mathbb{R}^n) = \{T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{R} \text{ (or } \mathbb{C}) \text{ linear and cts}\}$ the space of tempered distributions. These spaces are equipped with their weak-$*$ topologies of pointwise convergence on $\mathcal{D}, \mathcal{E}, \mathcal{S}$ resp. Have cts embeddings $\mathcal{E}' \subset \mathcal{D}'$, and $\mathcal{E}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$.

Example 5.4. Consider $\delta_x(\phi) = \phi(x)$, then if $\phi_j \to 0$ in $\mathcal{D}, \mathcal{E}, \mathcal{S}$, then $\delta_x(\phi_j) = \phi_j(x) \to 0$ as $j \to \infty$, so $\delta_x \in \mathcal{E}', \mathcal{D}', \mathcal{S}'$.

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Let $f \in L^1_{\text{loc}}(\Omega)$. Then $T_f(\phi) = \int_\Omega f\phi dx$, $\phi \in \mathcal{D}(\Omega)$. Have $T_f \in \mathcal{D}'(\Omega)$ since for $\phi_j \to 0$ in $\mathcal{D}(\Omega)$ we have $T_f\phi_j \to 0$ by DCT with dominating function $\sup_{j\in\mathbb{N}} \|\phi_j\|_\infty 1_K|f| \in L^1$ (K compact). Also, $T_f = 0$ in $\mathcal{D}'(\Omega)$ still implies $f = 0$ a.e. by applying the mollification theorem (ES) to $f1_{B(x)}$ where $B(x)$ is a ball in Ω containing $x \in \Omega$, so $L^1_{\text{loc}} \subseteq \mathcal{D}'$

If $\varphi_\varepsilon = \varepsilon^{-n}\varphi(\cdot/\varepsilon)$, $\varphi \geq 0$, smooth, compactly supported, normalized, then for $g \in \mathcal{D}(\mathbb{R}^n)$, $T_{\varphi_\varepsilon}(g) = g * \varphi_\varepsilon(0) \to g(0) = \delta_0(g)$ as $\varepsilon \to 0$. So $T_{\varphi_\varepsilon} \overset{\varepsilon\to 0}{\to} \delta_0$ in $\mathcal{D}'(\mathbb{R}^n)$.

5.1 Generalized (Distributional) Derivatives in $\mathcal{D}'(\Omega)$)

Let $f \in C^1(\Omega)$, then $D_i f \in L^1_{\text{loc}}$. Consider $T_{D_i f}$. Let $\varphi \in \mathcal{D}(\Omega)$

$$\int (D_i f)\varphi dx \overset{ibp}{=} -\int f D_i\varphi dx = -T_f(D_i\varphi)$$

So we define (for all multi-index α) the generalized derivative of any $T \in \mathcal{D}'(\Omega)$ as

$$(D^\alpha T)(\varphi) = (-1)^{|\alpha|} T(D^\alpha \varphi)$$

so $D^\alpha T \in \mathcal{D}'(\Omega)$. If $T = T_f$ and $D^\alpha T = T_g$ for some $f, g \in L^1_{\text{loc}}$, then say $g = D^\alpha_w f$ the weak partial derivative of f.

Example 5.5. Let $f(x) = x1_{\{x>0\}}$. Consider T_f.

$$DT_f(\varphi) = -T_f(\varphi') = -\int_0^\infty x\varphi'(x)dx \overset{ibp}{=} \int_0^\infty \varphi(x)dx = \int_{\mathbb{R}} H\varphi dx$$

where H is the Heaviside function, so H is the weak derivative of f.

Consider the second derivative.

$$D^2 T_f(\varphi) = DT_H(\varphi) = -T_H(\varphi') = -\int_0^\infty \varphi' = \varphi(0) = \delta_0(\varphi)$$

δ_0 cannot be represented by locally integrable functions.

Have $D^3 T_f(\varphi) = -\delta_0(\varphi') = -\varphi'(0)$ which is a Schwartz distribution but not a measure.

5.2 Multiplication of Distributions with Smooth Functions

If $f \in L^1_{\text{loc}}$ and $a \in C^\infty(\Omega)$, the $T_{af}(\varphi) = \int_\Omega af\varphi = T_f(a\varphi)$. Note that $a\varphi \in \mathcal{D}$ if $\varphi \in \mathcal{D}$, so we define

$$(aT)(\varphi) = T(a\varphi)$$

for $T \in \mathcal{D}'(\Omega)$ and $a \in C^\infty(\Omega)$.

5.3 Compactly Supported Distributions

Proposition 5.6. *A linear map $T : \mathcal{E}(\Omega) \to \mathbb{R}$ (or $\mathbb{C}$) is cts iff there exists $K \subseteq \Omega$ compact, $N \in \mathbb{N}$, and $C > 0$ s.t. for all $\varphi \in \mathcal{E}(\Omega)$*

$$|T(\varphi)| \leq C \max_{0 \leq |\alpha| \leq N} \sup_{x \in K} |D^\alpha \varphi(x)| \tag{†}$$

Proof. Suppose (†) holds and $\phi_j \to 0$ in $\mathcal{E}(\Omega)$. By defn of $\tau_{\mathcal{P}}$, for j large enough, $K \subseteq K_j$ and RHS of (†) with $\phi = \phi_j$ converges to 0, so $T(\phi_j) \to 0$, so $T \in \mathcal{E}'(\Omega)$.

Conversely, assume T is cts but (†) fails. If $K_j \subseteq K_{j+1}$ is any exhaustion of compact sets of Ω, we obtain a sequence $\varphi_j \in \mathcal{E}'(\Omega)$ s.t.

$$|T(\varphi_j)| \geq j \max_{0 \leq |\alpha| \leq j} \sup_{x \in K_j} |D^\alpha(\varphi_j)|$$

Define $\psi_j = \frac{\varphi_j}{|T(\varphi_j)|}$. Have

$$|D^\beta \psi_j(x)| \leq \frac{1}{j} \frac{|D^\beta \varphi_j(x)|}{\max_{0 \leq |\alpha| \leq j} \sup_{x \in K_j} |D^\alpha \varphi_j(x)|} \overset{e.v.}{\leq} \frac{1}{j} \to 0$$

So $\psi_j \to 0$ in $\mathcal{E}(\Omega)$ but $T(\psi_j) = 1$ for all j. Contradiction. □

Definition 5.7. Say $T \in \mathcal{D}'(\Omega)$ has support in a closed set $K \subseteq \Omega$ if $T(\varphi) = 0$ whenever $\varphi \in C_c^\infty(\Omega \backslash K) \subseteq C_c^\infty(\Omega)$.

The last proposition implies that $T \in \mathcal{E}'(\Omega)$ is supported in some compact subset of Ω. If $f \in L^1_{loc}$ s.t. $f = 0$ outside of a compact set, then T_f is compactly supported. If $T \in \mathcal{D}'(\Omega)$ is compactly supported, then so is $D^\alpha T$ for any α.

Proposition 5.8. *Any $T \in \mathcal{E}'(\Omega)$ restricts to $T \in \mathcal{D}'(\Omega)$ of compact support. Any $T \in \mathcal{D}'(\Omega)$ that is compactly supported extends to $\tilde{T} \in \mathcal{E}'(\Omega)$.*

Proof. The first claim follows from (†) in the preceding proposition. Conversely, if K is compact and supports T_1. Take $\xi \in C_c^\infty(\Omega)$ s.t. $\xi = 1$ on K and define $\tilde{T}(\varphi) = T(\xi\varphi)$, $\varphi \in \mathcal{E}(\Omega)$, which define an element of $\mathcal{E}'(\Omega)$. □

5.4 Convolutions of Distributions

Notation: Recall the shift operator $\tau_x g = g(\cdot - x)$. Let $\overset{\vee}{g} = g(-\cdot)$, and $\tau_x \overset{\vee}{g} = g(x - \cdot)$.

In this notation $f * g(x) = T_f(\tau_x \overset{\vee}{g})$. If $g \in C_c^\infty(\mathbb{R}^n)$, $\tau_x \overset{\vee}{g} \in C_c^\infty(\mathbb{R}^n)$, and we can define for $T \in \mathcal{D}'(\mathbb{R}^n)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$, the convolution

$$x \mapsto T * \varphi(x) = T[\tau_x \overset{\vee}{\varphi}]$$

Theorem 5.9. *Let $T \in \mathcal{D}'(\mathbb{R}^n)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$, α any multi-index. Then $T * \varphi \in C^\infty(\mathbb{R}^n)$ and $D^\alpha(T * \varphi) = (D^\alpha T) * \varphi = T * (D^\alpha \varphi)$.*

Proof. Take $e_i \in \mathbb{R}^n$ (basis vector), and let $h \to 0$. Write

$$\frac{1}{h}[T * \varphi(x + he_i) - T * \varphi(x)] = T\left[\frac{\varphi(x + he_i - \cdot) - \varphi(x - \cdot)}{h}\right]$$
$$\overset{ES2}{\longrightarrow} D_i(\varphi(x - \cdot))$$

in $\mathcal{D}(\mathbb{R}^n)$ and since T is cts in this topology, RHS $\overset{h \to 0}{\to} T[D_i\varphi(x - \cdot)] = T(\tau_x \overset{\vee}{D_i\varphi}) = T * D_i\varphi$. In particular $T * \varphi$ is cts for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and is $T * D_i\varphi$, and by iterating we deduce that $T * \varphi \in C^\infty(\mathbb{R}^n)$. In particular, $D^\alpha(T * \varphi) = T * (D^\alpha\varphi)$.

Need to prove the first equality. Note

$$D^\alpha(\tau_x \overset{\vee}{\varphi}) = D^\alpha \varphi(x - \cdot) = (-1)^{|\alpha|}(D^\alpha\varphi)(x - \cdot) = (-1)^{|\alpha|}\tau_x \overset{\vee}{D^\alpha}\varphi$$

Thus $(D^\alpha) * \varphi(x) = D^\alpha T(\tau_x \overset{\vee}{\varphi}) = (-1)^{|\alpha|}T(D^\alpha(\tau_x \overset{\vee}{\varphi})) = \tau(\tau_x \overset{\vee}{D^\alpha}\varphi) = T * (D^\alpha\varphi)$. ☐

Notice if $T \in \mathcal{E}'(\Omega)$, supported in K cpt and K_x the shifted support of $\tau_x \overset{\vee}{\varphi} = \varphi(x - \cdot)$, $\varphi \in C_c^\infty(\mathbb{R}^n)$. Thus for $|x|$ large enough, $K \cap K_x = \varnothing$ and $T_* \varphi \in \mathcal{D}(\mathbb{R}^n)$.

Definition 5.10. Let $T)1 \in \mathcal{D}'(\mathbb{R}^n)$, $T_2 \in \mathcal{E}'(\mathbb{R}^n)$. THen define their convolution by the action

$$(T_1 * T_2) * \varphi(x) = T_1 * (T_2 * \varphi)(x)$$

for $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

Remark 10. Note that $T_1 * T_2$ is assigned on $\mathcal{D}(\mathbb{R}^n)$ as we can consider $x = 0$, $\varphi = \phi(-\cdot)$ so that $T_1 * T_2(\varphi)(0) = T_1 * T_2(\phi)$ for $\phi \in \mathcal{D}(\mathbb{R}^n)$.

Remark 11. Note that δ_0 has cpt support and $\delta_0 * \varphi = \delta_0[\varphi(x - \cdot)] = \varphi(x)$. Therefore for any $T_1 \in \mathcal{D}'(\mathbb{R}^n)$, $(T_1 * \delta_0) * \varphi = T_1 * (\delta_0 * \varphi) = T_1 * \varphi$, so $\delta_0 * [\cdot]$ acts as a right identity on all of $\mathcal{D}'(\mathbb{R}^n)$.

Theorem 5.11. *Let $T_1 \in \mathcal{D}'(\mathbb{R}^n), T_2 \in \mathcal{E}'(\mathbb{R}^n)$, α any multi-index. Then*

$$D^\alpha(T_1 * T_2) = (D^\alpha T_1) * T_2 = T_1 * (D^\alpha T_2)$$

Proof. Using the previous theorem and the definitions, for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we have $D^\alpha(T_1 * T_2) \overset{\text{thm}}{=} (T_1 * T_2) * D^\alpha\varphi \overset{\text{def}}{=} T_1 * (T_2 * D^\alpha\varphi) \overset{\text{thm}}{=} T_1 * (D^\alpha T_2 * \varphi) \overset{\text{def}}{=} (T_1 * D^\alpha T_2) * \varphi$. ☐

5.5 Fundamental Solutions of Linear PDEs

Consider a partial differential operator $L = \sum_{|\alpha| \le k} a_\alpha D^\alpha$, $a_\alpha \in C^\infty(\mathbb{R}^n)$, $k \in \mathbb{N}$. Consider $Lu = u_0$ for $u, u_0 \in \mathcal{D}'(\mathbb{R}^n)$. A weak solution u is one s.t. $(Lu)(\varphi) = u_0(\varphi)$ for all $\varphi \in \mathcal{D}$. An element $\mathcal{G} \in \mathcal{D}'(\mathbb{R}^n)$ is called a fundamental solution for L if $L\mathcal{G} = \delta_0$ in $\mathcal{D}'(\mathbb{R}^n)$. If $\mathcal{G} = T_g$ for some $g \in L^1_{\text{loc}}(\mathbb{R}^n)$, then we call g the Green kernel of $\mathcal{G}$.

Theorem 5.12. *Suppose L has constant coefficients $a_\alpha \in \mathbb{R}$ (or $\mathbb{C}$), and $\mathcal{G} \in \mathcal{D}'(\mathbb{R}^n)$ is its fundamental solution. Then, if $u_0 \in \mathcal{E}'(\mathbb{R}^n)$, a solution $Lu = u_0$ is given by*

$$u = \mathcal{G} * u_0$$

Remark 12. If $u_0 \in \mathcal{D}$, then $\mathcal{G} * u_0 \in C^\infty(\mathbb{R}^n)$ and the equation $Lu = u_0$ holds pointwise on $\mathbb{R}^n$.

Proof. By linearity and the previous theorem

$$Lu = \sum_{|\alpha| \le k} a_\alpha D^\alpha(\mathcal{G} * u_0) = \sum_{|\alpha| \le k} a_\alpha(D^\alpha\mathcal{G} * u_0) = LG * u_0 = \delta_0 * u_0 = u_0$$

Note that the last equality follows from the fact that we can swap δ_0 and u_0 when u_0 is compactly supported. ☐

5.6 Fourier Transforms of Distributions

Recall Fourier transform

$$\hat{f}(u) = \int_{\mathbb{R}^n} e^{-ix\cdot u} f(x) dx$$

for $f \in L^1$. Since $\hat{f} \in L^1_{\text{loc}}(\mathbb{R}^n)$, we can consider the operator

$$T_{\hat{f}}(\varphi) = \int_{\mathbb{R}^n} \hat{f}(u)\varphi(u) du = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)e^{-ix\cdot u}\varphi(u) du dx = T_f(\hat{\varphi})$$

for $\varphi \in \mathcal{D}$. Note that we used Fubini. Since $\hat{\varphi}$ is not necessarily in $\mathcal{D}$, this defn doesn't extend to $\mathcal{D}'$, and we choose to work with $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ instead.

Recall Riemann Lebesgue lemma from PM.

Lemma 5.13. *Let $f \in L^1(\mathbb{R}^n)$, then $\hat{f} \in C_0(\mathbb{R}^n)$.*

Proof. For any $u_j \to u$ in $\mathbb{R}^n$, we have $e^{-ix\cdot u_j} f(x) \to e^{-ix\cdot u} f(x)$, and this gives a dominating function. By DCT, $\hat{f}(u_j) \to \hat{f}(u)$, so $\hat{f}$ is cts. Also have $\|\hat{f}\|_\infty \le \int_{\mathbb{R}^n} |f(x)| dx = \|f\|_{L^1}$. For any $f \in L^1$, take a sequence $f_k \in C_c^\infty(\mathbb{R}^n)$ s.t. $f_n \to f$ in L^1, so that $\|\hat{f}_k - \hat{f}\|_\infty \le \|f_k - f\|_{L^1} \to 0$ so $\hat{f}_k \to \hat{f}$ unif. on $\mathbb{R}^n$, and $\hat{f}_k \in C_0(\mathbb{R}^n)$. Now have $|u_j||\hat{f}_l(u)| = |\hat{D_j f_k}(u)| \le \|D_j f\|_{L^1} < \infty$. By completeness of $C_0(\mathbb{R}^n)$, $\hat{f} \in C_0(\mathbb{R}^n)$ □

Remark 13. Note that Fourier transform does not map L^1 onto $C_0(\mathbb{R}^n)$.

$\wr$; (Owen's signature)

Lemma 5.14. *Let $f \in L^1(\mathbb{R}^n)$.*

(i) If $f_\lambda = \lambda^{-n} f(\cdot/\lambda)$, $\lambda > 0$, then $\hat{f}_\lambda = \hat{f}(\lambda u)$, $u \in \mathbb{R}^n$

(ii) $\mathcal{F}[\tau_x f](u) = e^{-ix\cdot u}\hat{f}(u)$, $\mathcal{F}[e^{i\langle y,\cdot\rangle} f] = \tau_y \hat{f}$.

(iii) If $g \in L^1$, then $f * g \in L^1$ and $\mathcal{F}[f * g] = \hat{f}\cdot\hat{g}$

Proof. Fubini and substitution. □

Theorem 5.15.

(i) Let $f \in C^1(\mathbb{R}^n)$, $f, D_j f \in L^1$ for $j = 1, ..., n$. Then $\mathcal{F}[D_j f](u) = iu_j \hat{f}(u)$ for $u \in \mathbb{R}^n$.

(ii) If $f \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n}(1 + |x|)|f(x)| dx < \infty$, then for any $j = 1, ..., n$, $u \in \mathbb{R}^n$, have $D_j\hat{f}(u) = -i\mathcal{F}[x_j \hat{f}(x)]$. In particular, $\hat{f} \in C^1(\mathbb{R}^n)$

Proof. (i) For any $\varepsilon > 0$, we can pick $f_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ s.t. $\|f_\varepsilon - f\|_{L^1} + \|D_j f_\varepsilon - D_j f\|_{L^1} < \varepsilon$. [First approximate by $\bar{f} = f\xi$, with $\xi \in C_c^\infty(\mathbb{R}^n)$ s.t. $\xi = 1$ on $D(0, M)$. Approximate $\bar{f}$ by $\phi_\varepsilon * \bar{f} \to \bar{f}$ in L^1. Also have $D_j(\phi_\varepsilon * f) = \phi_\varepsilon * (D_j\bar{f}) \to D_j\bar{f}$ as $\varepsilon \to 0$.] For such f_ε we see

$$\mathcal{F}[D_j f_\varepsilon](u) = \int_{\mathbb{R}^n} e^{ix\cdot u} D_j f_\varepsilon(x) dx \overset{ibp}{=} -\int_{\mathbb{R}^n} iu_j x^{-ix\cdot u} f_\varepsilon(x) dx$$

So

$$\begin{aligned}|\mathcal{F}[D_j f](u) - iu_j\hat{f}(u)| &\le |\mathcal{F}[D_j\hat{f}](u) - \mathcal{F}[D_j f_\varepsilon](u)| + |iu_j(\hat{f}_\varepsilon(u) - f(u))| \\ &\le \|D_j f - D_j f_\varepsilon\| + |u_j|\|f_\varepsilon - f\|_{L^1} \\ &\le (1 + |u_j|)\varepsilon\end{aligned}$$

(ii)

$$\begin{aligned}\frac{1}{h}(\hat{f}(u + he_j) - \hat{f}(u)) &= \int_{\mathbb{R}^n} \frac{1}{h}(e^{-ix\cdot(u+he_j)} - e^{-ix\cdot u}) f(x) dx \\ &= \int_{\mathbb{R}^n} e^{-ix\cdot u}\left(\frac{e^{-ix\cdot(he_j)} - 1}{h}\right) f(x) dx \overset{\text{DCT}}{\to} -i\int_{\mathbb{R}^n} e^{-ix\cdot u} x_j f(x) dx\end{aligned}$$

Dominating function $|x_j|$, which is $|f(x)|dx$-integrable by assumption. □

Recall from PM.

Theorem 5.16 (Fourier Inversion). *Let $f \in L^1$ and $\hat{f} \in L^1$ Then $f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot u}\hat{f}(u)dx = \mathcal{F}^{-1}[\hat{f}](x)$ a.e.*

Note that for the unique cts representative of f, the formula holds everywhere.

Note that $\mathcal{F}^{-1}[\mathcal{F}\varphi] = \frac{1}{(2\pi)^n}\mathcal{F}[\mathcal{F}\varphi](-\cdot)$, so $\mathcal{F}^{-1}$ is a Fourier transform, and $\mathcal{F}^2\varphi = (2\pi)^n\overset{\vee}{\varphi}$.

Theorem 5.17. *$\mathcal{F}$ is a linear automorphism of $\mathcal{S}(\mathbb{R}^n)$.*

Proof. Can check $\mathcal{S}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$. If $f \in L^1$, then $\int |f| \leq (\sup_{x\in\mathbb{R}^n}(1+|x|)^{n+1}|f(x)) \int_{\mathbb{R}^n}\frac{dx}{(1+|x|)^{n+1}} < \infty$. For multi-indices α, β,

$$|u^\alpha||D^\beta \hat{f}(u)| = |\mathcal{F}[D^\alpha(x^\beta f)]|(u) \overset{\text{RL}}{\leq} \|D^\alpha(x^\beta f)\|_{L^1} \leq p_N(f)$$

where $p_N(f)$ is an expression of the form in the previous ineq. If $\phi_j \to 0$ in $\mathcal{S}$, have $p_N(\hat{\phi}_j) \to 0$ and $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is cts. Moreover, if $\mathcal{F}[\phi] = 0$ for $\phi \in \mathcal{S} \subseteq L^1$, then by the Fourier inversion formula, $\phi = \mathcal{F}^{-1}(0) = 0$, so F is injective. For any $\phi \in \mathcal{S}(\mathbb{R}^n)$, have $\mathcal{F}^{-1}\mathcal{F}\phi = \frac{1}{(2\pi)^n}\mathcal{F}^2\overset{\vee}{\phi}$, which is the fourier transform of some function, so also surjective. ☐

Definition 5.18. For $T \in \mathcal{S}'(\mathbb{R}^n)$ we define its distributional Fourier transform $\hat{T}(\phi) = T(\hat{\phi})$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$

Remark 14. Clearly by the previous theorem, $\hat{T} \in \mathcal{S}'(\mathbb{R}^n)$. If $f \in L^1$, then $T_{\hat{f}}(\phi) = \int_{\mathbb{R}^n}\hat{f}\phi dx \overset{\text{Fubini}}{=} \int_{\mathbb{R}^n} f\hat{\phi} = T_f(\hat{\phi})$.

(: ≀ (Owen's Signature with Quiff)

Definition 5.19. Call ϕ slowly increasing if $\sup_{x\in\mathbb{R}^n}(1+|x|)^{-N}|\phi(x)| < \infty$ for some N.

Then $T_\phi \in \mathcal{S}'(\mathbb{R}^n)$. Even if $\hat{T}_\phi$ is given by T_g for some $g \in L^1_{\text{loc}}$, can't conclude $\hat{\phi}$ is pointwise defined.

If $T_j, T \in \mathcal{S}'(\mathbb{R}^n)$ and $T_j \to T$ weak-* in $\mathcal{S}'(\mathbb{R}^n)$, then $\hat{T}_j(\phi) = T_j(\hat{\phi}) \to T(\hat{\phi}) = \hat{T}(\phi)$, so $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is sequentially continuous. One shows further that $\hat{T} = 0 \implies T = 0$, so $\mathcal{F}$ is inj. Define $\mathcal{F}^{-1}T$ via $\mathcal{F}^{-1}T(\phi) = T(\mathcal{F}^{-1}\phi)$ for all ϕ in $\mathcal{S}(\mathbb{R}^n)$. $\mathcal{F}^{-1} = \frac{1}{(2\pi)^n}\overset{\vee}{\mathcal{F}}$. Can check $\mathcal{F}^{-1}[\mathcal{F}T](\phi) = T(\phi)$.

Theorem 5.20. *$\mathcal{F}$ (Fourier transform) defines a linear automorphism of $\mathcal{S}'(\mathbb{R}^n)$.*

Remark 15. Recall Plancherel from PM. $\mathcal{F}$ extends to the completion by unif continuity. Get an isometry $\frac{1}{(2\pi)^{n/2}}\bar{\mathcal{F}}$ of $L^2 = \overline{L^1 \cap L^2}^{L^2}$. If we define $\bar{\mathcal{F}}T(\phi) = T(\bar{\mathcal{F}}\phi) = T(\hat{\phi})$ for all $\phi \in \mathcal{S} \subseteq L^1 \cap L^2$, then see that $\bar{\mathcal{F}} = \mathcal{F}$ on $\mathcal{S}^1$.

For any finite measure μ on $\mathbb{R}^n$, have $\hat{\mu}(u) = \int_{\mathbb{R}^n} e^{-ix\cdot u}d\mu(x)$. Then

$$\hat{T}_\mu(\phi) = T_\mu(\hat{\phi}) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot u}\phi(u)dud\mu(x) = T_{\hat{\mu}}(\phi)$$

$\mathcal{F}T_\mu = T_{\hat{\mu}}$ in $\mathcal{S}'(\mathbb{R}^n)$.

For $T \in \mathcal{E}'(\mathbb{R}^n)$, can define $E(u) = T(e^{-i\langle\cdot,u\rangle})$. Can show that $\hat{T} = T_E$ in $\mathcal{S}'(\mathbb{R}^n)$ with E slowly increasing.

Note that the product of a slowly increasing func with a rapidly decreasing func is again rapidly decreasing, i.e., in $\mathcal{S}$. For $T \in \mathcal{S}'$, define aT for $a \in C^\infty$ slowly increasing by $(aT)(\phi) = T(a\phi)$ for $\phi \in \mathcal{S}$. For any $T \in \mathcal{D}'(\mathbb{R}^n)$, define $(\tau_k T)(\varphi) = T(\tau_{-k}\phi)$ for $k \in \mathbb{R}^n$.

Lemma 5.21. *Let $T \in \mathcal{S}'(\mathbb{R}^n)$ and α any multi-index*

(i) $\mathcal{F}(\tau_y T) = e^{-i\langle y,\cdot\rangle}\hat{T}$ *and* $\mathcal{F}[e^{-i\langle y,\cdot\rangle}T] = \tau_y T$

(ii) $\mathcal{F}[D^\alpha T] = i^{|\alpha|}u^\alpha\hat{T}$ *and* $D^\alpha\hat{T} = (-i)^{|\alpha|}\mathcal{F}[x^\alpha T]$

Proof. Compute. ☐

Remark 16. $\mathcal{F}[D^\alpha\delta_0] = i^{|\alpha|}u^\alpha$. So the FT of partial derivatives of Dirac measure span the space of polys.

Owen is ill today

5.7 Periodic Distribution

Definition 5.22. $T \in \mathcal{D}'(\mathbb{R}^n)$ is periodic if $\tau_k T = T$ for all $k \in \mathbb{Z}^n$

Definition 5.23. For $T \in \mathcal{E}'(\mathbb{R}^n)$, define the periodization $T_{\text{per}} = \sum_{k\in\mathbb{Z}^n} \tau_k T$.

The fundamental cell of the lattice is $\mathcal{Q} = [-1/2, 1/2)^n$. The indicator $1_{\mathcal{Q}}$ is not smooth.

Lemma 5.24. *There exists* $\psi \in C_c^\infty(\mathbb{R}^n)$ *s.t.*

(i) $\psi \geq 0$

(ii) $\operatorname{supp}\psi \subseteq \operatorname{Int}(Q)$ *where* $Q = [-1,1]^n$.

(iii) $\sum_{k\in\mathbb{Z}^n} \psi(x-k) = 1$ *for all* $x \in \mathbb{R}^n$

If ψ' *is another such function and* T *is a periodic distribution, then* $T(\psi) = T(\psi')$.

Call this ψ a periodic partition of unity (ppu).

Proof. Find $\psi_0 \in C_0^\infty$ supported in $\operatorname{Int}(Q)$ s.t. $\psi_0 = 1$ on $\mathcal{Q}$. Define $S(x) = \sum_{k\in\mathbb{Z}^n} \psi_0(x-k)$. Normalize $\psi(x) = \psi_0(x)/S(x)$.

If T is a periodic distribution, then

$$T(\psi) = T(\sum_{g\in\mathbb{Z}^n} \tau_g\psi'\psi) = \sum_g \tau_g T(\psi\tau_g\psi') = \sum_g T(\psi'\tau_g\psi) = T(\psi')$$

□

Can take $\psi_{0,j} \to 1_{\mathcal{Q}}$ ptwise and $\sup_j \|\psi_{0,j}\|_\infty < \infty$. Obtain a uniformly bounded sequence of ppu $\psi_j \to 1_{\mathcal{Q}}$.

Definition 5.25. For $T \in \mathcal{D}'(\mathbb{R}^n)$ periodic, define the mean of T as $M(T) = T(\psi)$ where ψ is any ppu.

Theorem 5.26. *Let* $\mathcal{E}'(\mathbb{R}^n)$. T_{per} *converges in* $\mathcal{S}'(\mathbb{R}^n)$. *If* $T \in \mathcal{D}'(\mathbb{R}^n)$ *is periodic, then there exists* $V \in \mathcal{E}'(\mathbb{R}^n)$ *s.t.* $T = \sum_{g\in\mathbb{Z}^n} \tau_g V$ *in* $\mathcal{D}'(\mathbb{R}^n)$.

Proof. For $T \in \mathcal{E}'$, have a cpt set $K \subseteq B_R$ (ball of radius R) and $N, C > 0$ s.t. for all $\phi \in \mathcal{E}$

$$|T(\phi)| \leq C \sup_{x\in K, |\alpha|\leq N} |D^\alpha\phi(x)|$$

Have $1 + |g| \leq 1 + |g+x| + |x| \leq 1 + |g+x| + R \leq (1+R)(1+|g+x|)$, so

$$1 \leq \frac{(1+R)^M(1+|g+x|)^M}{(1+|g|)^M}$$

for any $M \in \mathbb{N}$. For all $\phi \in \mathcal{S} \subseteq \mathcal{E}$,

$$|T\phi| \leq C\frac{(1+R)^M}{(1+|g|)^M} \sup_{x\in K, |\alpha|\leq N} (1+|g+x|)^M |D^\alpha\phi(x)|$$

Applies to $\tau_g\phi$, get a similar inequality.

Since $\sum_{g\in\mathbb{Z}^n}(1+|g|)^{-n-1} < \infty$ we deduce

$$|\sum_g \tau_g T\phi| \leq C' \sup_{y\in\mathbb{R}^n, |\alpha|\leq N} (1+|y|)^{n+1}|D^\alpha\phi(y)|$$

so $\sum_g \tau_g T \in \mathcal{S}'$ by ES.

For the converse, let T be periodic and $\phi \in \mathcal{D}$. If ψ is any ppu, have

$$T\phi = T(\phi\sum_g \tau_g\psi) = \sum_g T(\psi\tau_{-g}\phi) = \sum_g (\psi T)(\tau_{-g}\phi) = \sum_g \tau_g(\psi T)(\phi)$$

□

(Owen's Signature)

Theorem 5.27 (Convergence of Fourier series in $\mathcal{S}'$)**.** *Let $U \in \mathcal{D}'(\mathbb{R}^n)$ be periodic. Then $U = \sum_{g\in\mathbb{Z}^n} u_g T_{e_{2\pi g}}$ in $\mathcal{S}'(\mathbb{R}^n)$, where $e_{2\pi h} = e^{i\langle 2\pi h, \cdot\rangle}$ and with Fourier coefficients $u_g = M(e_{2\pi h})$*

Lemma 5.28. *If $T \in \mathcal{S}'$ s.t. $(e_{-k} - 1)T = 0$ for all $k \in \mathbb{Z}^n$, then $T = \sum_{g\in\mathbb{Z}^n} c_g \delta_{2\pi g}$ in $\mathcal{S}'$. Have $|c_g| \le C(1+|g|)^N$ for some $N, C > 0$*

Proof. Let $\Lambda^* = \{2\pi g : g \in \mathbb{Z}^n\}$. Take $\varphi \in \mathcal{D}(\mathbb{R}^n)$ s.t. $\operatorname{supp}\varphi \cap \Lambda^* = \varnothing$, so $(e_{-k}-1)^{-1}\varphi \in \mathcal{D}$ and $T(\varphi) = (e_{-k}-1)T((e_{-k}-1)^{-1}\varphi) = 0$ for all k, so T is also supported in Λ^*. Now take ppu ψ and consider $\tilde{\psi} = \psi(\cdot/(2\pi))$. $\operatorname{supp}\tilde{\psi} \subset \{x \in \mathbb{R}^n : \forall i,\ -2\pi < x_i < 2\pi\}$ and $\sum_{g\in\mathbb{Z}^n} \tau_{2\pi g}\tilde{\psi} = 1$ on $\mathbb{R}^n$. Now define $T_g = (\tau_{2\pi g}\tilde{\psi})T$ which is supported in $\{2\pi g\}$ and have $\sum_{g\in\mathbb{Z}^n} T_g = \sum_g (\tau_{2\pi g}\tilde{\psi})T = T$ (in $\mathcal{D}'$) and $(e_{-k}-1)T_g = (\tau_{2\pi g}\tilde{\psi})(e_{-k}-1)T$ for all $k \in \mathbb{Z}^n$.

Choose $k = g_j$ (j-th standard basis vector), have $(e_{-k}-1)T_g = (e^{-ix_j}-1)T_g = (e^{-(x_j-2\pi g)}-1)T_g \overset{\text{Taylor}}{=} (x_j - 2\pi g)K(x_j)T_g$, where K is the Taylor poly which doesn't vanish near $2\pi g$, so $(x_j - 2\pi g)T_g = 0$.

Take $\phi \in \mathcal{S}(\mathbb{R}^n)$ and apply Taylor expansion to get $\phi(x) = \phi(2\pi g) + \sum_{j=1}^n (x_j - 2\pi g)\phi_j(x)$ for some $\phi_j \in \mathcal{S}$, so $T_g\phi = T_g(\phi(2\pi g)) + \sum_{j=1}^n (x_j - 2\pi g)T\phi_j = \delta_{2\pi g}(\phi)T_g(1)$. Let $c_g = T_g(1)$.

$|c_g| = |T_g(\sum_{g'} \tau_{2\pi g'}\tilde{\psi})| = |T_g(\tau_{2\pi g}\tilde{\psi})|$. Since $T_g \in \mathcal{E}' \subseteq \mathcal{S}'$ and we have a characterization of $\mathcal{S}'$ in ES3, have $|c_g| \le C_0 \sup_{x\in\mathbb{R}^n, |\alpha|\le N} (1+|x|)^N |D^\alpha \tilde{\psi}(x - 2\pi g)|$ for some $N \in \mathbb{N}$, $c > 0$, and $\le C_1(1+|x|)^N \sup_{y\in\mathbb{R}^n, |\alpha \le N|} (1+|y|)^N |D^\alpha\tilde{\psi}(y)| \le C(1+|g|)^N$. Therefore $T = \sum_g c_g\delta_{2\pi g}$ converges in $\mathcal{S}'$. □

Proof of Thm. Apply the lemma to U. $\hat{U} = (2\pi)^n \sum_{g\in\mathbb{Z}^n} u_g \delta_{2\pi g}$, $u_g = c_g/(2\pi)^n$. Take inverse FT, see $U = \sum_{g\in\mathbb{Z}^n} u_g T_{e_{2\pi g}}$. Note that $T \mapsto M[T]$ is cts on $\mathcal{S}'$, so $M(e_{-2\pi k}U) = \sum_{g\in\mathbb{Z}^n} u_g M(e_{-2\pi k}T_{e_{2\pi g}}) = \int_Q e^{i2\pi\langle g-k, x\rangle}dx = 1$ if $g = k$ and 0 otherwise. □

Apply this to $U = \sum_k \delta_k = \sum_k \tau_k\delta_0$ with ppu ψ s.t. $\psi(0) = 1$. Compute Fourier coeffs. $M(e_{-2\pi g}U) = \sum_k \delta_k(e_{-2\pi g}\psi) = 1$ for all g, so $\sum_{k\in\mathbb{Z}^n}\delta_k = \sum_{k\in\mathbb{Z}^n} T_{e_{2\pi k}}$ in $\mathcal{S}'$. Testing this identity on $\phi(x - \cdot)$ for $\phi \in \mathcal{S}$, $x \in \mathbb{R}^n$. Get $\sum_k \phi(x-k) = \sum_k T_{e_{2\pi k}}\phi(x-\cdot) = \sum_k e^{i2\pi k\cdot x}\hat{\phi}(2\pi k)$ (Poisson summation formula when $x = 0$).

[Owen Broke LaTeX Today]

6 Sobolev Spaces and Elliptic PDEs

Let $C^k(\Omega)$ denote th normed space $\{f : \Omega \to \mathbb{R} : D^\alpha f$ exists for all $0 \le |\alpha| \le k,\ \|f\|_{C^k} < \infty\}$, where $\|f\|_{C^k} = \sum_{0\le|\alpha|\le k}\|D^\alpha f\|_\infty$. Similarly define the Hölder spaces for $0 < \eta < 1$ as $C^{k,\eta}(\Omega) = \{f \in C^k(\Omega) : \|f\|_{C^{k,\eta}} < \infty\}$, where $\|f\|_{C^{k,\eta}} = \|f\|_k + \sum_{|\alpha|=k}\sup_{x\neq y}\frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^\eta}$. C^k and $C^{k,\eta}$ are Banach spaces.

We can replace $\|\cdot\|_\infty$ and D^α by L^p-norm and the weak derivative D_w^α.

Definition 6.1 (Sobolev space)**.** Let $k \in \mathbb{Z}_{\ge 0}$, $1 \le p \le \infty$. Then $f \in W^{k,p}(\Omega)$ (Ω open) if $D_w^\alpha f \in L^p$ for all $0 \le |\alpha| \le k$. Then norm on $W^{k,p}(\Omega)$ is given by

$$\|f\|_{W^{k,p}} = \left(\sum_{0\le|\alpha|\le k}\|D_w^\alpha f\|_{L^p}^p\right)^{1/p}$$

if $p < \infty$ and

$$\|f\|_{W^{k,\infty}} = \max_{0\le|\alpha|\le k}\|D_w^\alpha f\|_{L^\infty}$$

When $\Omega = \mathbb{R}^n,\ p = 2$, have

Definition 6.2. Let $s \in \mathbb{R}$. Then $H^s(\mathbb{R}^n)$ consists of $f \in \mathcal{S}'(\mathbb{R}^n)$, $(f = T_f)$, s.t. $\hat{f} \in L^2_{\text{loc}}(\mathbb{R}^n)$ and $\|f\|_{H^s}^2 = \int_{\mathbb{R}^n}|\hat{f}(u)|^2(1+|u|^2)^s du$

Note that $H^s(\mathbb{R}^n)$ is a Hilbert space for the inner product $(f,g)_{H^s} = \int_{\mathbb{R}^n}\hat{f}(u)\overline{\hat{g}(u)}(1+|u|^2)^s du$, so $H^s = L^2(\mu_s)$ for some measure μ_s on $\mathbb{R}^n$.

By Plancherel, for $s \ge 0$, $H^s(\mathbb{R}^n)$ consists of elements of $L^2(\mathbb{R}^n, dx)$

Proposition 6.3. *For $s \geq 0$, $H^s(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n)$ with equivalent norms*

Proof. □

Theorem 6.4 (Sobolev embedding)**.** *Let $s > n/2 + k$ for $k \in \mathbb{N}$ and $f \in H^s$. Then $\exists f^* \in C^k(\mathbb{R}^n)$ s.t. $f^* = f$ a.e. and $\|f^*\|_{C^k} \leq C_{s,n,k}\|f\|_{H^s}$. In particular, there is an embedding $H^s(\mathbb{R}^n) \hookrightarrow C^k(\mathbb{R}^n)$.*

Remark 17.

$\{\varnothing\}$: Have $H^s \subseteq C^{k,\eta}$ if $s > \frac{n}{2} + k + \eta$.

$\{\varnothing, \{\varnothing\}\}$: $\bigcap_{s>0} H^s \subseteq C^\infty(\mathbb{R}^n)$

Proof. Take $f \in \mathcal{S}(\mathbb{R}^n)$ and note

$$
\begin{aligned}
|D^\alpha f(x)| &= |\mathcal{F}^{-1}[u^\alpha \hat{f}]| \\
&\overset{\text{R.L.}}{\leq} \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} |u|^{|\alpha|}|\hat{f}(u)|\frac{(1+|u|^2)^{s/2}}{(1+|u|^2)^{s/2}}du \\
&\overset{\text{C.S.}}{\leq} \frac{1}{(2\pi)^n}\left(\int_{\mathbb{R}^n}\frac{|u|^{2|\alpha|}}{(1+|u|^2)^s}du\right)\left(\int_{\mathbb{R}^n}|\hat{f}(u)|^2(1+|u|^2)^s du\right) \\
&\leq C_{s,n,k}\|f\|_{H^s}
\end{aligned}
$$

For $f \in H^s$ take $f_n \in \mathcal{S}$ s.t. $f_n \to f$ in H^s and a.e. (pass to a subseq if necessary). This is Cauchy in H^s and by the same inequality in C^k, we have $f_n \to f^*$ in C^k by completeness. By uniqueness of limit, we have $f^* = f$ a.e. so $f^* = f$. □

Consider

$$-\nabla^2 v + v = f \tag{†}$$

where $f \in H^s(\mathbb{R}^n)$. Have Fourier transform $\hat{\nabla}^2 = -|u|^2$.

Theorem 6.5. *There exists a unique solution v in $H^{s+2}(\mathbb{R}^n)$ to (†) and $\|v\|_{H^{s+2}} \leq \|f\|_{H^s}$ (elliptic regularity estimate)*

Proof. Take FT get $(1+|u|^2)\hat{v} = \hat{f}$ in $\mathcal{S}'(\mathbb{R}^n)$. For $f \in L^1_{\text{loc}}$ this has unique soln $\hat{v}(u) = \frac{\hat{f}(u)}{1+|u|^2}$, $u \in \mathbb{R}^n$, so $v = \mathcal{F}^{-1}\hat{v}$.

$$\|v\|^2_{H^{s+2}} = \int_{\mathbb{R}^n}(1+|u|^2)^{s+r}\frac{\hat{f}(u)}{(1+|u|^2)^2} = \|f\|^2_{H^s}$$

□

To study eqns restricted to open sets $\Omega \subseteq \mathbb{R}^n$ with boundary $\partial\Omega$, need to define the restriction of $f \in H^s$ to $\partial\Omega$. If $f \in H^s$ for $s > n/2$, then Sobolev embedding implies that $f \in C^\varepsilon$ for some $\varepsilon > 0$ and the Sobolev trace $f|_{\partial\Omega}$ exists by uniform continuity. For general $s > 1/2$, have

Theorem 6.6 (Trace thm)**.** *There exists a bounded linear operator $T : H^s(\mathbb{R}^n) \to H^{s-1/2}(\mathbb{R}^{n-1})$, $s > 1/2$, s.t. for all $f \in \mathcal{S}(\mathbb{R}^n)$, $Tf = f|_{\mathbb{R}^{n-1}\times\{0\}}$*

Proof. ES □

Call $T = T_\Sigma$ for $\Sigma = \mathbb{R}^{n-1}\times\{0\}$ the boundary trace of $f \in H^s$. By change of coords, this operator extends to $T_{\partial\Omega}$ for sufficiently regular Ω. In particular, we have $T_{\partial\Omega} : H^1(\mathbb{R}^n) \hookrightarrow H^{1/2}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$ is bounded linear.

6.1 $H^1_0(\Omega)$

Any $f \in C_c^\infty(\Omega)$ (sufficiently regular Ω) extends by zero to an element of $H^1(\mathbb{R}^n)$ (hence in $H^s(\mathbb{R}^n)$ for all s). Have Hilbert norm

$$\|f\|_{H^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n}(1+|u|^2)||\hat{f}(u)|^2 du = (2\pi)^n\int_\Omega(|f(x)|^2 + |Df(x)|^2|)dx$$

where Df is the gradient vector. Define $H^1_0(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{H^1}}$ in $H^1(\mathbb{R}^n)$. This is not $W^{1,2}$ because

Proposition 6.7. *Let $f \in H_0^1(\mathbb{R}^n)$. Then $f(x) = 0$ for almost every $x \in \Omega^c$ and if $\partial\Omega$ is sufficiently regular, then $T_{\partial\Omega} f = 0$.*

Proof. Take $\varphi \in C_c^\infty((\Omega^c)^\circ)$ and take $f_n \in C_c^\infty(\Omega)$ s.t. $f_n \to f$ in $H^1(\mathbb{R}^n)$. Have $\Lambda_\varphi(h) = \int_{\mathbb{R}^n} \varphi h$, then $\Lambda_\varphi \in (L^2)' \subseteq (H^1)'$, so $0 = \int_{\mathbb{R}^n} \varphi f_n = \Lambda_\varphi(f_n) \to \Lambda_\varphi f = \int \varphi f = 0$, so $\operatorname{supp}(f) \subseteq \Omega$. Similarly, $0 = T_{\partial\Omega} f_n \to T_{\partial\Omega} f = 0$, so $f = 0$ on $\partial\Omega$. □

Consider the BVP

$$\begin{cases} -\nabla^2 v + v = f & \text{on } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

Interpret this as

$$\int_\Omega (-\nabla^2 v + v)\varphi \overset{ibp}{=} \int_{\partial\Omega} \nabla v \cdot \nabla\varphi + \int_\Omega v\varphi = \int_\Omega f\varphi$$

for $f \in L^2$, $v \in H^1$.

Since C_c^∞ is dense in H_0^1 and L^2, this equation is the same as solving

$$\langle v, \varphi \rangle_{H^1} = \langle f, \varphi \rangle_{L^2} \tag{$\dagger'$}$$

for all $\varphi \in H_0^1(\Omega)$.

Theorem 6.8. *For eveyr $f \in L^2(\Omega)$, there exists a unique $v \in H_0^1(\Omega)$ s.t. ($\dagger'$) holds and $\|v\|_{H^1} = \|f\|_{L^2}$. Therefore the solution map $S : f \mapsto v = v_f$ is a bounded linear form $L^2(\Omega) \to H_0^1(\Omega)$ and self-adjoint for $L^2(\Omega)$.*

Proof. Define $\Lambda_f(\phi) = \int_\Omega f\phi$ so that $\Lambda_f \in (H_0^1)'$ since

$$|\Lambda_f(\phi)| \overset{\text{C.S.}}{\leq} \|f\|_{L^2}\|\phi\|_{L^2} \leq \|f\|_{L^2}\|\phi\|_{H^1}$$

Hence by Riesz representation thm on H_0^1, there exists a uniuque $v \in H_0^1$ s.t. $\langle v, \phi \rangle_{H^1} = \langle f, \phi \rangle_{L^2}$ for all $\phi \in H_0^1$.

Next take $f_1, f_2 \in L^2(\Omega)$, $\alpha \in \mathbb{R}$, and take $v_1 = S(f_1)$ and $v_2 = S(f_2)$ and define $v = v_1 + \alpha v_2$. Then,

$$\langle v, \phi \rangle_{H^1} = \langle v_1 + \alpha v_2, \phi \rangle_{H^1} = \langle v, \phi \rangle_{H^1} + \alpha \langle v_2, \phi \rangle_{H^1} == \langle f_1, \phi \rangle_{L^2} + \alpha \langle f_2, \phi \rangle_{L^2} = \langle f_1 + \alpha f_2, \phi \rangle_{L^2}$$

So $S(f_1) + \alpha S(f_2) = S(f_1 + \alpha f_2)$. Also have

$$\|S(f)\|_{H^1} = \|v_f\|_{H^1} \overset{\text{Riesz}}{=} \|\Lambda_f\| \leq \|f\|_{L^2}$$

To see it's self-adjoint,

$$\langle S(f), g \rangle_{L^2} = \langle g, S(f) \rangle_{L^2} = \langle S(f), S(g) \rangle_{H^1} = \langle S(g), S(f) \rangle_{H^1} = \langle f, S(g) \rangle_{L^2}$$

□

To study regularity of v, we introduce

$$H^s_{\text{loc}}(\Omega) = \{f \in L^2_{\text{loc}}(\Omega) : \forall \xi \in C_c^\infty(\Omega),\ f\xi \in H^s(\mathbb{R}^n)\}$$

Proposition 6.9. *If $f \in H^s_{loc}$ for $s > k + n/2$, then $f \in C^k(U)$ for any U open s.t. $\bar{U} \subseteq \Omega$.*

Proof. Given U, pick $\xi \in C_c^\infty$ s.t. $\xi = 1$ on $\bar{U}$ and note that $f\xi \in H^s(\mathbb{R}^n) \subseteq C^k(\mathbb{R}^n)$ (Sobolev embedding), so $f = f\xi$on U, the result then follows. □

Corollary 6.10. $\bigcap_{s>0} H^s_{loc}(\Omega) \subseteq C^\infty(\Omega)$.

Note that $f \in C^\infty(\Omega)$ may be unbounded at $\partial\Omega$.

Theorem 6.11 (Interior regularity). *Let $f \in L^2(\Omega)$ and suppose $v \in H_0^1$ solves ($\dagger'$). Then $v \in H^2_{loc}(\Omega)$. If additionally $f \in L^2(\Omega) \cap H^k_{loc}(\Omega)$, then $v \in H^{k+2}_{loc}(\Omega)$.*

Proof. Let $K \subseteq \Omega$ be any compact set, and take $\chi \in C_c^\infty(\Omega)$ s.t. $\chi = 1$ on K. Take $\varphi \in S(\mathbb{R}^n)$ and set $\phi = \chi\varphi \in H_0^1$. Then $(\dagger')$ implies

$$\int_\Omega (Dv \cdot D(\varphi\chi) + v\varphi\chi)dx = \int_\Omega f\varphi\chi dx$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Using chain rule and IBP, we rearrange the equation above to get

$$\int_\Omega (D(v\chi) \cdot D\varphi + v\chi\varphi)dx = \int_\Omega g\varphi dx$$

where $g = -(Dv)\cdot(D\chi) - vD\chi + f\chi \in L^2(\mathbb{R}^n)$. IBP again, can see that $v\chi$ solves $-\nabla^2(v\chi) + v\chi = g$ in $\mathcal{S}'(\mathbb{R}^n)$. Hence by elliptic regularity estimate, $\|v\chi\|_{H^2} \leq \|g\|_{L^2} < \infty$.

To prove $v \in H^2_{\text{loc}}$, take $\xi \in C_c^\infty(\Omega)$ and $K = \text{supp}(\xi)$ s.t. $v\xi = v\chi\xi$. Then $\|v\xi\|_{H^2} = \|v\chi\xi\|_{H^2} \leq C_n\|v\chi\|_{H^2}\|\xi\|_{L^2} < \infty$ ($\|fg\|_{L^2} \leq \|f\|_{L^2}\|g\|_{L^\infty}$ + chain rule) We recognize that $g \in H^1_{\text{loc}}(\Omega)$ whenever $f \in H^1_{\text{loc}}(\Omega)$ so repeating the preceding argument given $v \in H^3_{\text{loc}}$. Can prove the rest of the theorem using the inequality $\|fg\|_{H^s} \leq C_{n,s}\|f\|_{H^s}\|g\|_{C^s}$ □

Corollary 6.12. *If $f \in C^\infty(\Omega)\cap L^2(\Omega)$, then $v \in H_0^1(\Omega)\cap C^\infty(\Omega)$ solves $-\nabla^2 v + v = f$ on Ω (pointwise)*

Theorem 6.13 (Rellich-Kondrashov). *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. Let $u_j \in H_0^1(\Omega)$ s.t. $\|u_j\|_{H^1} \leq K$ for all $j = 1, 2, \ldots$, and some $K > 0$. Then $\exists u \in H_0^1(\Omega)$ s.t. $u_{j_k} \to u$ in $L^2(\Omega)$ along a subsequence.*

Proof. By Banach-Alaoglu in $H_0^1(\Omega)$, we obtain $u_{j_k} \rightharpoonup u$ in H_0^1 and then in L^2 (weakly), and $\|u\|_{H^1} \leq K$. Also u_{j_k}, u vanish a.e. on Ω^c so

$$\begin{aligned}\|u_{j_k} - u\|_{L^2(\Omega)} = \|u_{j_k} - u\|_{L^2(\mathbb{R}^n)} &= \frac{1}{(2\pi)^n}\|\hat{u}_{j_k} - \hat{u}\|_{L^2(\mathbb{R}^n)} \\ &= \frac{1}{(2\pi^n)}\left(\int_{|z|>R} |\hat{u}_{j_k}(z) - \hat{u}(z)|^2 dz + \int_{|z|<R} |\hat{u}_{j_k}(z) - \hat{u}(z)|^2 dz\right)\end{aligned}$$

Given $\varepsilon > 0$, have

$$\int_{|z|>R} |\hat{u}_{j_k}(z) - \hat{u}(z)|^2 dz \leq \int_{|z|>R} \frac{1+|z|^2}{1+|z|^2}(|\hat{u}_{j_k}(z)|^2 + |\hat{u}(z)|^2)dz \leq \frac{2}{1+R^2}(\|u_{j_k}\|^2_{H^1} + \|u\|^2_{H^1}) < \varepsilon$$

for R sufficiently large.

For $z \in \mathbb{R}^n$ fixed,

$$\hat{u}_{j_k}(z) = \int_{\mathbb{R}^n} e^{-ix\cdot z} u_{j_k(x)} dx = \int_\Omega e^{-ix\cdot z} u_{j_k}(x) dx = \langle e^{-i\langle\cdot, z\rangle}, u_{j_k}\rangle \to \langle e^{-i\langle\cdot,z\rangle}, u\rangle_{L^2(\Omega)} = \hat{u}(z)$$

by weak convergence. Also

$$|\hat{u}_{j_k}(z)| + |\hat{u}(z)| \leq \|u_{j_k}\|_{L^1(\Omega)} + \|u\|_{L^1(\Omega)} \overset{\text{C.S.}}{\leq} C_\omega(\|u_{j_k}\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}) \leq 2C_\Omega K$$

which is dz-integrable on $\{z : |z| \leq R\}$, so $\int_{|z|<R} |\hat{u}_{j_k}(z) - \hat{u}(z)|^2 dz \to 0$ by DCT. □

Corollary 6.14. *The solution operator S from $(\dagger')$ is a compact linear self-adjoint operator on $L^2(\Omega)$*

Proof. S maps L^2 into H_0^1 (bounded linear) and use Rellich-Kondrashov. □

By spectral theorem, there exists ONB $\{w_k : k \in \mathbb{N}\}$ of $L^2(\Omega)$ and real e-values $\mu_k \downarrow 0$ as $k \to \infty$ s.t.

$$Sw_k = \mu_k w_k$$

in $L^2(\Omega)$. Thus $w_k \in H_0^1$. For all $\varphi \in H_0^1$

$$\langle w_k, \varphi\rangle_{L^2} \overset{(\dagger')}{=} \langle Sw_k, \varphi\rangle_{H^1} = \mu_k\langle w_k, \varphi\rangle_{H^1}$$

Tes $\varphi = w_k$, see that $1 = \langle w_k, w_k\rangle_{L^2} = \mu_k \|w_k\|_{H^1}$, so $\mu_k > 0$ for all k. In $\mathcal{D}'(\Omega)$,

$$(-\nabla^2 + I)w_k = (-\nabla^2 + I)\frac{\mu_k}{\mu_k} w_k = \frac{1}{\mu_k}(-\nabla^2 + I)Sw_k = \frac{w_k}{\mu_k}$$

Therefore,

$$-\nabla^2 w_k = \left(\frac{1}{\mu_k} - 1\right) w_k = \lambda_k w_k$$

where $\lambda_k = \frac{1}{\mu_k} - 1 \uparrow \infty$ are the e-values of $-\nabla^2$. The weak form is

$$\langle Dw_k, D\varphi\rangle_{L^2} = \lambda_k \langle w_k, \varphi\rangle_{L^2}$$

for all k and all $\varphi \in \mathcal{D}(\Omega)$ (in fact $H_0^1(\Omega)$) Note that $w_k = f$ in ($\dagger'$) and in H_0^1, so iterating the interior regularity thm, $w_k \in C^\infty(\Omega)$. Thus $-\nabla^2 w_k = \lambda_k w_k$ is true on Ω ptwise.

Theorem 6.15 (Poincare inequality)**.** *For all $u \in H_0^1(\Omega)$,*

$$\frac{\langle Du, Du\rangle_{L^2}}{\langle u, u\rangle_{L^2}} \geq \lambda_1 > 0$$

Proof. ES □

We can now solve the Dirichlet problem for the Laplace equation

$$\begin{cases} -\nabla^2 v = f & \text{on } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

or the weak form: find $v \in H_0^1$ $\langle Dv, D\varphi\rangle_{L^2} = \langle f, \varphi\rangle_{L^2}$ for all $\varphi \in \mathcal{D}(\Omega)$. Denote this by $(*)$

Theorem 6.16. *There exists a unique solution $v \in H_0^1(\Omega)$ to $(*)$, for any $f \in L^2(\Omega)$.*

$$v = \sum_{k=1}^\infty \frac{1}{\lambda_k}\langle w_k, f\rangle_{L^2} w_k$$

Proof. Take partial sums $v_J = \sum_{n=1}^J \lambda_k^{-1}\langle w_k, f\rangle_{L^2} w_k$, $J \in \mathbb{N}$, then (for $J' < J$)

$$\begin{aligned}
\|v_J - v_{J'}\|_{H^1}^2 &= \langle v_J - v_{J'}, v_J - v_{J'}\rangle_{L^2} + \langle D(v_J - v_{J'}), D(v_J - v_{J'})\rangle \\
&= \sum_{k=J'+1}^J \lambda_k^{-1}\langle w_k, f\rangle^2 + \sum_{k,k'=J'+1}^J \lambda_k^{-1}\lambda_{k'}^{-1}\langle f, w_k\rangle\langle f, w_{k'}\rangle\langle Dw_k, Dw_{k'}\rangle_{L^2} \\
&= \sum_{k=J'+1}^J \lambda_k^{-1}\langle w_k, f\rangle^2 + \sum_{k,k'=J'+1}^J \lambda_k^{-1}\lambda_{k'}^{-1}\langle f, w_k\rangle\langle f, w_{k'}\rangle\lambda_k^{-1}\langle w_k, w_{k'}\rangle_{L^2} \\
&\leq \sum_{k=J'+1}^\infty (\lambda_k^{-2} + \lambda_k^{-1}\langle w_k, f\rangle_{L^2}^2 \\
&\leq C(\lambda_1)\sum_{k=J'+1}^\infty \langle f, w_k\rangle^2 \overset{J'\to\infty}{\longrightarrow} 0
\end{aligned}$$

So v_J is Cauchy in H_0^1, so $v \in H_0^1$ Can check

$$\begin{aligned}
\langle Dv, D\varphi\rangle_{L^2} &\overset{ibp}{=} \langle v, -\nabla^2\varphi\rangle_{L^2} = \sum_{k=1}^\infty \lambda_k^{-1}\langle v, w_k\rangle_{L^2}\langle w_k, -\nabla^2\varphi\rangle_{L^2} \\
&= \sum_{k=1}^\infty \langle v, w_k\rangle_{L^2}\langle w_k, \varphi\rangle_{L^2} \\
&= \langle v, \varphi\rangle_{L^2}
\end{aligned}$$

We used that $\langle w_k, -\nabla^2\varphi\rangle_{L^2} = \langle Dw_k, D\varphi\rangle_{L^2} = \lambda_k\langle w_k, \varphi\rangle_{L^2}$.

Need to show uniqueness. Suppose $v' \in H_0^1(\Omega)$ s.t. $(*)$ holds. Then let $w = v - v' \in H_0^1(\Omega)$ where $\langle Dw, D\varphi\rangle_{L^2} = \langle f - f, \varphi\rangle_{L^2} = 0$ for all $\varphi \in H_0^1(\Omega)$. Now $\|w\|_{H^1}^2 = \langle w, w\rangle_{L^2} + \langle Dw, Dw\rangle_{L^2} \leq (\frac{1}{\lambda_1} + 1)\langle Dw, Dw\rangle_{L^2} = 0$, so $w = 0$ a.e. □

Remark 18. One can also show interior regularity estimates to deduce that for $f \in C^\infty(\Omega) \cap L^2(\Omega)$ then $v \in C^\infty(\Omega)$.

7 Variational Problems* (non-examinable)

Consider minimizing a functional $F(u) = \|u\|_{H^2}^2 - \langle f, u\rangle_{L^2}$ over $H_0^1(\Omega)$, where f is fixed in $L^2(\Omega)$

Theorem 7.1. *Let $f \in L^2(\Omega)$. Then $\inf\{F(u) : u \in H_0^1(\Omega)\} \geq \sigma > -\infty$. Moreover, there exists a unique $w \in H_0^1$ s.t. $F(w) = \sigma$, and w solves the PDE $-\nabla^2 w + w = f$ in the weak sense.*

Proof. Use the inequality $ab \leq a^2/2 + b^2/2$ for $a, b \geq 0$. By Cauchy-Schwarz, $F(u) = \|u\|_{H^1}^2 - 2\langle f, u\rangle_{L^2} \geq \|u\|_{H^1}^2 - 2\|f\|_{L^2}\|u\|_{L^2} \geq \|u\|_{H^1}^2 - 2\|f\|_{L^2}^2 - \frac{1}{2}\|u\|_{L^2}^2 \geq \frac{1}{2}\|u\|_{H^1}^2 - 2\|f\|_{L^2}^2 \geq -2\|f\|_{L^2} > -\infty$.

Then take $u_k \in H_0^1$ s.t. $F(u_k) \to \sigma$. WLOg, assume $|F(u_k)| \leq \bar{F} < \infty$. Then, $\|u_k\|_{H^1}^2 = F(u_k) + 2\langle f, u_k\rangle_{L^2} \leq \bar{F} + 2\|f\|_{L^2}\|u_k\|_{L^2} \leq \bar{F} + 2\|f\|_{L^2}^2 + \frac{1}{2}\|u_k\|_{H^1??}^2$. Subtracting we see $\|u_k\|_{H^1}^2 \leq 2\bar{F} + 4\|f\|_{L^2}^2$ for all k, so u_k is bounded in H_0^1. By Banach-Alaoglu, there exists $u_{k_j} \rightharpoonup w$ weakly in H_0^1 and L^2 for some $w \in H_0^1(\Omega)$. By sheet 2, know that $\|w\|_{H^1}^2 \leq \liminf \|u_{k_j}\|_{H^1}^2$ and $\langle f, w\rangle = \lim\langle f, u_{k_j}\rangle_{L^2}$. We have $F(w) = \|w\|_{H^1}^2 - 2\langle f, w\rangle_{L^2} \leq \liminf(\|u_{k_j}\|_{H^1} - 2\langle f, u_{k_j}\rangle_{L^2}) = \sigma$, and $F(w) \geq \sigma$ by defn of inf, so $F(w) = \sigma$.

To prove uniqueness, it suffices to show that w solves the PDE. For all $v \in H_0^1$, $t \in \mathbb{R}$, we have $F(w) \leq F(w + tv)$ and $\frac{d}{dt}F(w + tv)|_{t=0} = 0$. Then $F(w + tv) = \|w + tv\|_{H^1}^2 - 2\langle f, w + tv\rangle_{L^2} = \|w\|_{H^1}^2 + t^2\|v\|_{H^1}^2 + 2t\langle w, v\rangle_{H^1} - 2t\langle f, v\rangle_{L^2} - 2\langle f, w\rangle_{L^2}$, so $\frac{d}{dt}(F(w+tv)) = 2t\|v\|_{H^1} + 2(\langle w, v\rangle_{H^1} - \langle f, v\rangle_{L^2})$. At $t = 0$, must have $\langle w, v\rangle_{H^1} = \langle f, v\rangle_{L^2}$ for all $v \in H_0^1$. □