Linear Analysis

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1 Normed spaces & linear operators

Definition 1.1. Let X be a vector space over $\mathbb{R}$ or $\mathbb{C}$ (usually over $\mathbb{R}$). A **norm** on X is a function $\| \bullet \| : X \to \mathbb{R}_{\geq 0}$ s.t.

1. $\|x\| = 0$ iff $x = 0$;
2. $\|\lambda x\| = |\lambda| \|x\|$ for all $x \in X$, λ scalar;
3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ (Triangle inequality)

A **normed space** is a pair $(X, \| \bullet \|)$, where X is a vector space and $\| \bullet \|$ is a norm.

Example 1.2.
1. l_p^n, $X = \mathbb{R}^n$, $\|x\| = \|x\|_p = \left(\sum_i |x_i|^p\right)^{1/p}$. (Triangle inequality follows from Minkowski's inequality)
2. l_∞^n, $X = \mathbb{R}^n$, $\|x\| = \|x\|_\infty = \max_i |x_i|$.

Remark 1. Often useful to consider the unit ball $B = B(X)$. B completely determines the norm by $\|x\| = \inf\{t > 0 : x \in tB\}$. B is always convex by triangle inequality. In fact, can check that if $B \subseteq \mathbb{R}^n$ is a closed, bounded, convex, symmetric ($x \in B \implies -x \in B$), and is a nbd of 0, then B defines a norm via the formula above, and B is the unit ball of that norm.

Let S be the set of scalar sequences made into a vector space under pointwise operations.

Example 1.3.
1. $l_p = \{x \in S\colon \sum_i |x_i|^p < \infty\}$ with norm $\|x\| = \|x\|_p = \left(\sum_i |x_i|^p\right)^{1/p}$.
2. $l_\infty = \{x \in S : \sup_n |x_n| < \infty\}, \|x\|_\infty = \sup_n |x_n|$.
3. $C_0 = \{x \in S : x_n \to 0\}$ with norm $\| \bullet \|_\infty$.

Definition 1.4. A Banach space is a complete normed space.

1.1 The Hölder & Minkowski Inequality

Lemma 1.5. For $1 \leq p < \infty$, $x \mapsto x^p$ is convex.

Theorem 1.6 (Minkowski's inequality)**.** *Let $1 \leq p < \infty$, $x, y \in l_p$, then $x + y \in l_p$ and $\|x + y\|_p \leq \|x\|_p + \|y\|_p$.*

Proof. First assume $\|x\|_p, \|y\|_p \leq 1$. For each n, we have

$$|\lambda x_n + (1 - \lambda) y_n|^p \leq \lambda |x_n|^p + (1 - \lambda) |y_n|^p$$

by the preceding lemma. So we have the same inequality of partial sums. Taking limit, we see that $\lambda x + (1 - \lambda) y \in l_p$ and $\|\lambda x + (1 - \lambda) y\|_p \leq \lambda \|x\|_p + (1 - \lambda) \|y\|_p$. For general $x, y \in l_p$, wlog assume $x, y \neq 0$ as otherwise the inequality is trivial. Apply the aove to the normalized vectors, we get

$$\left\| \frac{\|x\|_p}{\|x\|_p + \|y\|_p} \frac{x}{\|x\|_p} + \frac{\|y\|_p}{\|x\|_p + \|y\|_p} \frac{y}{\|y\|_p} \right\|_p \leq 1$$

, which implies the claimed inequality after some manipulation. □

Definition 1.7. $1 \leq p < \infty$, the conjugate index to p is the q s.t. $1 < q < \infty$ with $1/p + 1/q = 1$.

Lemma 1.8. If p, q are conjugate, and $a, b \geq 0$, then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$.

Proof. Can either do Legendre transform or substitute $x = a^p$, $y = b^p$ and use concavity of log. □

Theorem 1.9 (Hölder's inequality)**.** *If p, q are conjugate, $x \in l_p$, $y \in l_q$, then $(x_n y_n) \in l_1$ with $\|(x_n y_n)\|_1 \leq \|x\|_p \|y\|_q$.*

Proof. WLOG, assume $\|x\|_p = \|y\|_q = 1$. For all n, $|x_n y_n| \leq \frac{|x_n|^p}{p} + \frac{|y_n|^q}{q}$, so $(x_n y_n) \in l_1$ with $\|x_n y_n\| \leq \frac{\|x\|_p^p}{p} + \frac{\|y\|_q^q}{q} = 1/p + 1/q = 1$. □

Example 1.10.

- $C([0,1])$ with the sup norm $\|f\|_\infty$ is complete (uniform convergence). More generally, $C(K)$ is with sup norm is complete if K is compact Hausdorff.
- $C([0,1])$ with the 1-norm $\|f\|_1 = \int_0^1 f$ is incomplete. Still incomplete with p-norm $(\int_0^1 |f|^p)^{1/p}$.
- $C^1([0,1])$ of ctsly diff function with the sup norm $\| \bullet \|_\infty$ is incomplete. However with a new norm $\|f\|' = \|f\|_\infty + \|f'\|_\infty$, the space is complete (exercise)

The following result suggests that for a norm to be complete, one often needs to take into account the structure of the space.

$\Delta = D^1 \subseteq \mathbb{C}$ the closed unit disk. $A(\Delta)$ be the space of functions which are cts on Δ and analytic on Δ°, then $A(\Delta)$ equipped with $\|f\|_\infty$ is complete (c.f. complex analysis, Montel's theorem, the uniform limit of holomorphic functions is holo'c).

Remark 2. In l_p, $(1 \leq p \leq \infty)$. Write $e_n = (0, ..., 0, 1, 0, ...)$ for the vector which is 1 at the nth place. The linear span of $\{e_n\}$ is $F = \{x \in l_p : \exists N \geq 0, \forall n \in N, x_n = 0\} \subsetneq l_p$. However, we get close for $1 \leq p < \infty$ since the closed linear span (the closure) of e_n is l_p. To see this note that $\forall x \in l_p$, $\sum_{n=1}^k x_n e_n \to x$ as $k \to \infty$.

The property above is false for l_∞, e.g., take $(1, 1, ...)$. In fact the closed linear span in this case is C_0.

Also note that Subspaces need not be closed, e.g. $F \subseteq l_1$ (even if F is dense).

Definition 1.11. A topological space X is separable if it has a countable dense subset.

Example 1.12. l_p $(1 \leq p < \infty)$ is separable (take finite rational sequences). l_∞ is not separable (exercise).

1.2 Linear Operators

Definition 1.13. X, Y normed, a linear map $T : X \to Y$ is an operator if it's cts.

Proposition 1.14. *X, Y normed, $T : X \to Y$ linear, then TFAE*

1. *T cts*
2. *T cts at 0*
3. *$\exists k$ s.t. $\|T\| \leq k\|x\|$ for all $x \in X$. (T is bounded)*

Proof. 1 implies 2 is trivial. If 2 is true, then let $B(Y)$ be the unit ball in Y, which is a nbd of 0, so by continuity at 0, there exists $\delta > 0$ s.t. $\|x\| < \delta \implies \|Tx\| \leq 1$. Rearranging, get $\|Tx\| \leq \|x\|/\delta$ for all $x \in B(Y)$, but then linearity allows us to say the same thing for all $x \in X$. To prove 3 implies 1, we simply note that Lipschitz condition implies continuity. □

Definition 1.15. For bounded $T : X \to Y$, the operator norm of T is $\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\}$.

Then for all x, $\|Tx\| \leq \|T\|\|x\|$.

We write $L(X, Y)$ for the set of cts linear maps $X \to Y$. In some sources, the notation $B(X, Y)$ is used.

Remark 3. One can check that the operator norm is indeed a norm on $L(X, Y)$.

Proposition 1.16. *If $S : X \to Y$, $T : Y \to Z$ are operators, then $T \circ S$ is an operator and $\|TS\| \leq \|T\|\|S\|$.*

Proof. Direct computation. □

Example 1.17. 1. $T : l_p^n \to l_p^n$ defined by projecting onto the first two coords. $\|T\| = 1$.

2. Right shift. This is an injective isometry, but not surjective.

3. Left shift. This is surjective but not injective.

4. p, q conj. For $x \in l_p, y \in l_q$, write $x.y$ for $\sum x_n y_n$ which conv. abs. by Hölder. Can then define $\phi_y : l_p \to \mathbb{R}$ by $x \mapsto x.y$. Have $\phi_y \in L(l_p, \mathbb{R})$ and $\|\phi_y\| \le \|y\|_q$.

5. Take $(F, \|\bullet\|_1)$ and deifne $T : F \to \mathbb{R}$, $x \mapsto \sum nx_n$. T is linear but not cts.

6. $T : l_1 \to l_2$, $x \mapsto x$. Then $\|T\| = 1$ as $\sum |x_n| \le 1 \implies \forall n, |x_n| \le 1 \implies \sum |x_n|^2 \le 1 \implies \|T\| \le 1$. T is not surjective but the image contains F, so $\mathrm{im}(T)$ is dense. It is not closed so incomplete.

Definition 1.18. X, Y normed.

- The dual $X^* = L(X, \mathbb{R})$ with operator norm
- $X \to Y$ is an isomorphism if it's a linear homeo, i.e., $\exists c, d > 0$ s.t. $c\|x\| \le \|Tx\|$ and $\|Tx\| \le d\|x\|$.
- $X \to Y$ is an isometric isomorphism if it's an isometry.
- For isomorphic X, Y, the Banach-Mazur distance from X to Y is $d(X, Y) = \inf_{T \text{ iso}} \|T\|\|T^{-1}\|$.
- Two norms are equivalent if they induce the same topology, i.e., the identity is a homeo, i.e., two norms are Lipschitz equivalent. In particular, if two norms are equiv, and one of them is complete, then both are complete.

Note that for equivalent norms, $c\|x\|_1 \le \|x\|_2 \le d\|x\|_1$, i.e., $d^{-1}B_1 \subset B_2 \subset c^{-1}B$.

Remark 4.
- On $C[0,1]$, $\|\bullet\|_\infty$ is complete, $\|\bullet\|_1$ is incomplete, so they are not equiv. The map $\mathrm{id} : (C[0,1], \|\bullet\|_\infty) \to (C[0,1], \|\bullet\|_1)$ is a cts linear bijection whose inverse is not cts.
- On F, $\|\bullet\|_1$ and $\|\bullet\|_2$ are not equiv. Take, $x = e_1 + ... + e_n$, then $\|x\|_1 = n$, $\|x\|_2 = \sqrt{n}$.
- (T_n) a sequence in $L(X, Y)$ and $T_n \to T$ in operator norm, then $T_n \to T$ pointwise. The converse is false: $T_n : l_1 \to \mathbb{R}, x \mapsto x_n$. Then $\|T_n\| = 1$ for all n, but we do have pointwise convergence.

Theorem 1.19. *X, Y normed, Y complet. Then $L(X, Y)$ is complete with respect to the operator norm.*

Proof. Given a Cauchy sequence (T_n) in $L(X, Y)$, Pointwise, have $\|T_n x - T_m y\| \le \|T_n - T_m\|\|x\|$, so pointwise Cauchy. Since Y is complete, the pointwise limit exists. Can check that the pointwise limit T is a linear map. It suffices to prove boundedness. Given $\epsilon > 0$, there exists N s.t. $\|T_m - T_n\| \le \epsilon$ for all $m \ge n \ge N$. Let $m \to \infty$, we get $\|Tx - T_n x\| \le \epsilon\|x\|$ for all x and $n \in N$. So for all x, have

$$\|Tx\| \le (\epsilon + \|T_m\|)\|x\|$$

Note that $\|T_m\|$ is bounded as the sequence is Cauchy, so we have a uniform bound, so $T \in L(X, Y)$. Now, the preceding argument essentially says that $\|T - T_n\| \le \epsilon$ for all $n \ge N$. □

Corollary 1.20. *If X is a complete norm space, then X^* is complete.*

We want to investigate the dual of l_p. Let $1 < p, q < \infty$ be conjugate. For each $y \in l_q$, we defined $\phi_y \in l_p^*$. We know that $\|\phi_y\| \le \|y\|_q$. We construct $x_n = \mathrm{sgn}(y_n)|y_n|^{q/p}$, then $(x_n) \in l_p$ with $\|x\|_p = \|y\|_q^{q/p}$. $\phi_y(x) = \sum |y_n|^{q/p+1} = \sum |y_n|^q = \|y\|_q$. We see that $\|\phi_y\| = \|y\|_q$.

Theorem 1.21. *Let $1 < p, q < \infty$ be conjugate. Then $\theta : l_q \to l_p^*, y \mapsto \phi_y$ is an isometric isomorphism.*

Proof. This map is obviously linear. It's norm preserving so obviously continuous and injective. To show surjectivity, consider $T \in l_q^*$ and define the following sequences (**important trick!**) $y_n = Te_n$.

$$x_n = \begin{cases} \mathrm{sgn}(y_n)|y_n|^{q/p} & n \le N \\ 0 & \text{o/w} \end{cases}$$

Then $x \in l_p$ and $Tx = \sum_1^N |y_n|^{q/p+1} = \sum_1^N |y_n|^q$, so

$$\sum_1^N |y_n|^q \le \|T\| \|x\|_p = \|T\| \left(\sum_1^N |y_n|^q \right)^{1/p}$$

so

$$\|T\| \ge \left(\sum_1^N |y_n|^q \right)^{1/q}$$

Let $N \to \infty$, we see that $y \in l_q$. By linearity and continuity, $T = \phi_y$ on $\overline{F} = l_q$. □

Remark 5.
1. Same argument shows that $l_1^* = l_\infty$ and $C_0^* = l_1$ using the density of $\langle e_n : n \in \mathbb{N} \rangle$. However, it wouldn't show $l_\infty^* = l_1$. In fact, there exists $T \in l_\infty^*$ not of the form ϕ_y. The proof of this fact is beyond the scope of this course.
2. Each l_p $(1 \le p \le \infty)$ is complete and each is a dual space
3. Cannot have $X^* = \{0\}$ by Hahn-Banach theorem (Any cts functional on a subspace extend to a continuous functional on the whole space).

1.3 Finite Dimensional Spaces

Theorem 1.22. *Any two norms on a finite dimensional vector space are equivalent.*

Proof. Will show that any norm on $\mathbb{R}^2$ is equiv to $\| - \|_\infty$. For $x \in \mathbb{R}^n$, $\|x\| \le \sum \|x_i e_n\| = \sum |x_i| \|e_i\| \le n(\max_i \|e_i\|)\|x\|_\infty$.

Conversely, consider $f : S \to \mathbb{R}$, where $S = \{x \in \mathbb{R}^n : \|x\|_\infty = 1\}$, then f is continuous on compact set so there eixsts $\delta > 0$ s.t. $\|x\|_\infty = 1 \implies \|x\| \ge \delta$, so if $0 \ne x \in \mathbb{R}^n$, then $x = \|x\|_\infty \hat{x}$ and $\|x\| \ge \delta \|x\|_\infty$. □

Corollary 1.23. *X, Y normed and* $\dim X < \infty$, *then every lienar map $T : X \to Y$ is cts.*

Proof. Define norm on X yb $\|x\|' = \|x\| + \|Tx\|$, so there exists c s.t. $\|x\|' \le c\|x\|$ by equivalence of norm, so $\|Tx\| \le c\|x\|$ by unwinding a bit. □

Corollary 1.24. *X, Y finite dim normed spaces of with* $\dim X = \dim Y$. *Then X, Y are isomorphic.*

Corollary 1.25.
1. *Y normed finite dim vec space, then Y complete;*
2. *X normed, then if $Y \subset X$ is a finite dim subspace then Y is closed.*

Corollary 1.26. *If X is a finite dim normed vec space then B_x is compact (being closed and bounded in 2-norm).*

Note that the unit ball in l_p is not compact since $\|e_i - e_j\| \ge 1$ for all i, j.

Define $d(x, Y) = \inf\{d(x, y) : y \in Y\}$ as the distance from x to a closed subspace Y. Note that $d(x, Y) = 0$ iff $x \in Y$ (closed).

Proposition 1.27 (Riesz's lemma)**.** *If X is normed and Y a proper closed subspace of X, then for all $\epsilon > 0$, there exists $x \in X$ with $\|x\| = 1$ s.t. $d(x, Y) \ge 1 - \epsilon$. Moreover, if* $\dim X < \infty$, *then $\exists x \in X$ s.t. $\|x\| = 1$ with $d(x, Y) = 1$.*

Proof. Let $\epsilon > 0$ be given. Choose $x \in X$ s.t. $x \in Y$ and $\|x\| = 1$. Pick $y \in Y$ with $\|x - y\| \le (1 + \epsilon) d(x, Y)$. Define $z = \frac{x-y}{\|x-y\|}$. Can check that for any $y' \in Y$, we have

$$z - y' = \frac{x - y - \|x - y\| y'}{\|x - y\|} \ge \frac{d(x, Y)}{\|x - y\|} \ge \frac{1}{1 + \epsilon}$$

For the second part, note that $d(x, Y)$ is a cts function $B_X \to \mathbb{R}$. The domain is compact if X is finite dimensional, so attains its lower bound. □

Theorem 1.28. *X infinite dim normed space. Then there eixsts a seq. (x_n) in X with $\|x\| = 1$ for all n and $\|x_n - x_m\| \ge 1$ for all $m \ne n$. In particular, B_X is not compact.*

Proof. Choose $x_1 \in X$ with unit norm. Inductively set $Y = \langle x_1, ..., x_n \rangle$ and $X' = \langle x_1, ..., x_n, x \rangle$ for any $x \notin Y$. By Riesz's lemma, there exists $x_{n+1} \in X'$ s.t. $\|x_{n+1}\| = 1$ and $d(x_{n+1}, Y) = 1$. □

1.4 Compact Operators

Definition 1.29. X, Y normed. A linear operator $T : X \to Y$ is compact if $\overline{T(B_X)}$ is compact.

Example 1.30. For instance, if T has finite rank, then $\overline{T(B_X)}$ is closed and bounded in finite dimensional space hence compact.

If X is infinite dimensional, then id_X is not compact due to the preceding theorem.

Remark 6.

1. Compact implies cts.
2. Note that if Y is complete, then T is compact if and only if $T(B_X)$ is totally bounded.
3. $T : X \to Y$ is compact iff for any (x_n) in B_X, there exists a subsequence (x_{n_i}) with $T(x_{n_i})$ convergent.

Proposition 1.31. *X, Y normed. Y complete. Then $\{$compact operators$\}$ is a closed subspace of $L(X, Y)$.*

Proof. Take a sequence (x_n) in B_X. then can find a subsequence whose image under S converges. Pass to some subsequence of this subsequence so that the image under T also converges, so if S, T are compact then so is $S + T$. This implies that the space of compact operators form a subspace.

Consider a sequence $T_n \to T$ for T_n compact. Given $\epsilon > 0$, choose n with $\|T_n - T\| < \epsilon$. Since $T_n(B_X)$ is totally bounded, we have $T_n(B_X) = \bigcup_{i=1}^m B(T_n(x_i), \epsilon)$ for some x_i. So, $T(B_X) \subseteq \bigcup_i B(T_n(x_i), 2\epsilon)$ and $T(B_X) \subseteq \bigcup_i B(T(x_i, 3\epsilon))$, which gives total boundedness. □

In particular, the limit of finite rank operators is compact.

Example 1.32. For fixed $1 \le p \le \infty$, the projection onto the nth coord p_n is a finite rank operator with $\|p_n\| = 1$. $p = \sum_n p_n/n^2$ converges as it's Cauchy, then p is compact but not of finite rank.

Proposition 1.33. *X, Y, Z normed, $S \in L(X, Y)$, $T \in L(Y, Z)$, then*

1. *S compact $\implies T \circ S$ compact*
2. *T compact $\implies T \circ S$ compact.*

Proof. To prove the first claim, consider a sequence (x_n) in B_X, then there is a subsequence with Sx_{n_i} convergent, so TSx_{n_i} is convergent by continuity.

To prove the second claim. Note that for any sequence (x_n) in B_X, Sx_n is bounded, so TSx_{n_i} converges for some subsequence as T is compact. □

We have seen the map $T : l_1 \to l_2$ with dense image but not all of l_2. For instance $x(1/\sqrt{n}, ..., 1/\sqrt{n}, 0, 0, ...)$ then $\|x\|_2 = 1$ $\|x\|_1 = \sqrt{n}$.

Theorem 1.34 (Open mapping lemma)**.** *X, Y normed. X complete, then $T \in L(X, Y)$. Suppose $\overline{T(B_X)} \supset B_Y$, then*

1. *$\forall y \in Y, \exists x \in X$ s.t. $Tx = y$ and $\|x\| \le 2\|y\|$ (In particular, T is surjective.)*
2. *Y complete.*

Remark 7.

1. $\overline{T(B_X)} \supset B_Y$ implies that the image of B_X is dense in B_Y, i.e., $\forall y \in B_Y, \forall \epsilon > 0$, $\exists x \in B_X$ with $\|Tx - y\| < \epsilon$.
2. The proof would also show that for all $y \in Y$, there eixsts x with $\|x\| \le |1 + \epsilon|\|y\|$ for any fixed ϵ.
3. We say that T is open if the image of any open set is open, i.e., $T(B_X)$ is a nbd of 0, i.e., $T(B_X) \supset \frac{1}{k} B_Y$ for some $k \in \mathbb{N}$, i.e., $\forall y \in Y, \exists x \in X$ with $Tx = y$ and $\|x\| \le k\|y\|$. Therefore, open mapping lemma says that if $\mathrm{im}(B_X)$ is desne in B_Y, then T is open.

Proof.

1. Given $y \in Y$ with unit norm. Seek $x \in X$ with $Tx = y$ and $\|x\| \le 2$. $T(B_X)$ is dense in B_Y, so $\exists x_1 \in X$ s.t. $\|x_1\| \le 1$ and $\|y - Tx_1\| \le 1/2$. Also, $T(\frac{1}{2} B_X)$ is dense in $1/2 B_Y$, so $\exists x_2 \in X$ with $\|x_2\| \le 1/2$ s.t. $\|y - T(x_1 + x_2)\| \le 1/4$. Continue, obtain $x = \sum x_n$ which is Cauchy, hence convergent, and $\|y - Tx\| = 0$, so $Tx = y$.
2. Given (y_n) Cauchy in Y. WLOG assume $\|y_n - y_{n-1}\| \le 2^{-n}$ (by passing to a subsequence if necessary). For each $n \ge 2$, choose $x_n \in X$ s.t. $\|x_n\| \le 2^{-n+1}$ with $Tx_n = y_n - y_{n+1}$. Also, choose $x_1 \in X$ with $Tx_1 = y_1$ (surjectivity). Thus $T(x_1 + ... + x_n) = y_n$. Put $x = \sum x_n$ which is Cauchy hence convergent. Then $Tx = \lim_n T(\sum_1^n x_i) = \lim y_n$, so y_n converges.

□

1.5 New Spaces From Old

Definition 1.35 (Direct sum)**.** $X \oplus_p Y$ $(1 \le p \le \infty)$, where $\|(x,y)\|_p = (\|x\|^p + \|y\|^p)^{1/p}$ and $\|(x,y)\|_\infty = \sup(\|x\|, \|y\|)$.

These norms are all equivalent, so we usually write $X \oplus Y$. If X, Y are Banach then so is $X \oplus Y$. X, Y are always closed subspaces of $X \oplus Y$.

Definition 1.36 (Quotient)**.** X normed, $N \le X$ a closed linear subspace. We can define the quotient space X/N. Define $\|z\| = \inf\{\|x\| : x \in X, \pi(x) = z\}$, where $\pi : X \to X/N$ is the canonical projection map (continuous in this norm). If X is Banach, then so is X/N.

Proof. The only non-trivial property to check is positivity. If $\|z\| = 0$ for some $z \in X/N$, then there exists $x_1, x_2, ... \in X$ with $\pi(x_n) = z$ and $\|x_n\| \to 0$. Then let $\pi(x) = z$, we have $x_n \to 0$ and $x_n - x \in N$, so $x \in N$ as N is closed, so $z = 0$.

Clear that the projection is cts as $\|\pi(x)\| \le \|x\|$ by definition.

If X is Banach, then $\forall z \in X/N$ with $\|z\| \le 1$, there exists $x \in X$ with $\|x\| < 1$ with $\pi(x) = z$, so $\pi(B_X)$ is dense in $B_{X/N}$. Done by open mapping lemma. □

Definition 1.37 (Completion)**.** X normed, there exists a completion of X defined as

$$\tilde{X} = \{\text{Cauchy sequences in } X\}/\sim$$

where $(x_n) \sim (y_n)$ iff $x_n - y_n \to 0$. This is naturally a vector space equipped with the norm $\|[(x_n)]\|_{\tilde{X}} = \lim \|x_n\|$. Then $\tilde{X}$ is complete and X is a dense subspace of $\tilde{X}$ given by embedding $x \mapsto (x, x, x, ...)$.

Warning: 1st iso fails, e.g., $l_1 \to l_2, x \mapsto x$ is injective, but the image is not complete.

2 Baire Category Theorem

Theorem 2.1 (Baire Category Theorem)**.** *X non-empty complete metric space and $O_1, O_2, ...$ a sequence of dense open sets. Then $\bigcap_{n \ge 0} O_n \ne \varnothing$.*

Proof. $O_1 \ne \varnothing$, so $\exists \overline{B(x_1, \epsilon_1)} \subseteq O_1$ for some x_1 and $\epsilon < 1$. O_2 is dense, so can find $\overline{B(x_2, \epsilon)} \subseteq B(x_1, \epsilon_1) \cap O_2$ for some x_2 and $\epsilon_2 < 1/2$. Continue, get a nested sequence of closed balls. We have $\epsilon_n \to 0$ and $\overline{B(x_n, \epsilon_n)} \subseteq O_n$. (x_n) is Cauchy so converges by completeness, so $x_n \to x \in \bigcap_n O_n$. □

Remark 8. Cannot omit "dense" (e.g., $(0,1), (2,3)$ in $\mathbb{R}$) or "open" (e.g. $\mathbb{Q}, \mathbb{R} \setminus \mathbb{Q} \subseteq \mathbb{R}$), or "complete" (e.g. $O_n = \mathbb{Q} \setminus \{q_n\}$, where $q_1, ...$ is an enumeration of rationals in $\mathbb{Q}$.)

The exact same proof shows that $\bigcap_n O_n$ is dense in X.

Note that O open dense iff O^c closed and has empty interior.

Theorem 2.2 (BCT')**.** *X non-empty complete metric space. $F_1, F_2, ...$ closed in X with $\bigcup_n F_n = X$ Then some X has non-empty interior.*

Definition 2.3. $A \subset X$ (metric) is nowhere dense (ND) if it's not dense in any open ball, i.e., $A^\circ = \varnothing$.

It is clear that A ND iff $\overline{A}$ ND. If A is closed, then A ND iff A contains no open ball.

Theorem 2.4 (BCT")**.** *X non-empty complete metric space. $A_1, ...$ ND subsets of X, then $\bigcup_n A_n \ne X$.*

Definition 2.5. $A \subset X$ (metric) is meager if $A = \bigcup_n A_n$, A_n ND for all n. (Countable union of ND subsets)

Then BCT" implies that X is not a meager subset of X.

Note that there exists uncountable ND subset of $[0, 1)$, i.e., the Cantor set $\{\sum_n a_n 3^{-n} : a_n = 0 \text{ or } 2\}$.

2.1 Applications of BCT

Proposition 2.6 (Osgood's Theorem)**.** *If (f_n) is a sequence of cts functions on $[0,1]$ which is pointwise bounded, then there exists $a < b$ s.t. f_n is unif. bounded on $(a,b) \subseteq [0,1]$.*

Proof. Define $E_n = \{x \in [0,1] : \forall i, |f_i(x)| \le n\}$, then $\bigcup_n E_n = [0,1]$ and E_n is closed for all n, so some E_n has non-empty interior. □

Theorem 2.7 (Principle of uniform boundedness)**.** *X Banach, Y normed. Let $T_1, T_2, ..., \in L(X,Y)$. If (T_n) is pointwise bounded. Then T_n are uniformly bounded.*

Proof. Define $E_n\{x \in X : \forall i, \|T_i(x)\| \le n\}$ (closed, cover X), so there exists $B(x,\epsilon) \subseteq E_n$ for some n, x, ϵ. So for any $y \in X$ s.t. $\|y\| < \epsilon$, $x, x+y \in B(x,\epsilon)$. So $\forall i$, $\|T_i y\| = \|T_i(x+y) - T_i(x)\| \le 2n$, so $\|T_i\| \le 2n/\epsilon$. □

Theorem 2.8 (Banach-Steinhaus Theorem)**.** *X Banach, Y normed, $T_1, T_2, ... \in L(X,Y)$ s.t. $T_n \to T$ pointwise. Then $T \in L(X,Y)$ (linear and cts)*

Proof. Linearity is trivial. Need to show continuity. $\forall x \in X$, $T_n x \to Tx$ so T_n is pointwise bounded hence uniformly bounded by principle of uniform boundedness. So $\exists M$ s.t. $\|T_n\| \le M$ for all n, so $\|T_n x\| \le \|T_n\|\|x\| \le M\|x\|$. Let $n \to \infty$, we see that $\|Tx\| \le M\|x\|$. □

Theorem 2.9 (Open Mapping Theorem)**.** *X, Y Banach, $T \in L(X,Y)$ surjective. Then T is an open map [$\exists R$ s.t. $\forall y \in Y$ $\exists x \in X$ s.t. $Tx = y$ with $\|x\| \le R\|y\|$.]*

Proof. T is surjective, so $\bigcup_n \overline{T(nB_X)} = Y$ which each $\overline{T(nB_X)}$ closed. Baire $\implies$ that some $T(nB_X)$ contains a ball $B(y,\epsilon)$. WLOG, $n=1$, so $T(B_X)$ is dense in $B(y,\epsilon)$. Now for all $z \in Y$ with $\|z\| < \epsilon$, we have $y+z, y-z \in B(y,\epsilon)$, so there exists $x, x' \in B_X$ with $\|Tx - (y+z)\| < \delta$ and $\|Tx' - T(y-z)\| < \delta$, which implies that $\frac{x-x'}{2} \in B_X$ and $\|T((x-x')/2) - z\| < \delta/2 + \delta/2 = \delta$. So $T(B_X)$ is dense in $B(0,\epsilon)$. By open mapping lemma (applied to the operator T/ϵ), there exists $x \in X$ with $Tx = \epsilon y$ with $\|x\| \le 2\|y\|$, i.e., $T(x/\epsilon) = y$ and $\|x/\epsilon\| \le \frac{2}{\epsilon}\|y\|$. □

Corollary 2.10 (Inversion Theorem)**.** *X, Y Banach. $T \in L(X,Y)$ bijective, Then T is an isomorphism*

Proof. T surjective, so T is open by open mapping thm, so $\exists k$ s.t. $\forall y \in Y$, $\|T^{-1}y\| \le k\|y\|$, so T^{-1} is cts. □

Corollary 2.11 (Comparability theorem)**.** *Let $\| \|_1$, $\| \|_2$ be complete norms on V. If $\exists c > 0$ s.t. $\forall x$, $\|x\|_2 \le c\|x\|_1$. Then these two norms are equivalent.*

Proof. Note that $\mathrm{id} : (V, \| \|_1) \to (V, \| \|_2)$ is a continuous bijection. Done by inversion theorem. □

Theorem 2.12 (Closed graph theorem (CGT))**.** *X, Y Banach, $T \in L(X,Y)$. Then T is cts iff* Graph(T) *is closed.*

Remark 9. This has an important consequence. T has closed graph $\Leftrightarrow$ If $(x_n, Tx_n) \to (x,y)$, then $y = Tx$. $\Leftrightarrow$ If $x_n \to x$ and $Tx_n \to y$, then $y = Tx$. $\Leftrightarrow$ If $x_n \to 0$, $Tx_n \to y$, then $y = 0$.

CGT $\implies$ to show T cts, it suffices to show that if $x_n \to 0$ and $Tx_n \to y$ (can assume the existence of this limit) then $y = 0$.

Proof. "$\Rightarrow$": Trivial.

"$\Leftarrow$": Let $G(T)$ denote the graph of T. $G(T)$ is closed in $X \times Y$ (with $\| \|_1$ say) by assumption, so $G(T)$ is complete. Define $S : X \to G(T)$, $x \mapsto (x, Tx)$ WTS that S is bounded. Clearly S is a bijection and S^{-1} is cts ($\|x\| \le \|x\| + \|Tx\|$), so we are done by inversion theorem. □

2.1.1 Existence of a cts nowhere differentiable function

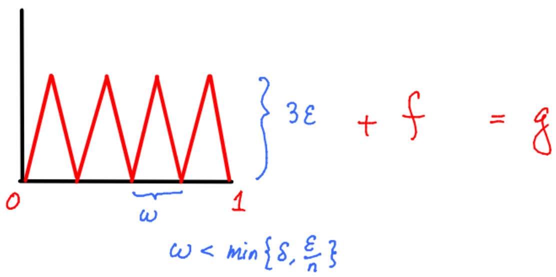
$f \in C[0,1]$ is diffrentiable at x iff $\frac{f(x+t)-f(x)}{t} \to 0$ as $t \to 0$. In particular the difference quotient is bounded for all $t \neq 0$ small. Define $E_n = \{f \in C[0,1] : \exists x, \ \forall t, \ |(f(x+t) - f(x))/t| \leq n\}$

Claim: E_n is closed: Suppose $(f_i) \in E_n$ with $f_i \to f$ unif. For each i, there exists x_i s.t. $|(f_i(x_i+t) - f_i(x_i))/t| \leq n$ for all t. By passing to a convergent subsequence if necessary, we may assume $x_i \to x$ as $i \to \infty$ for some $x \in [0,1]$. Then have

$$\frac{f_i(x_i+t) - f_i(x_i)}{t} \to \frac{f(x+t) - f(x)}{t}$$

for all t. Thus, $|(f(x+t) - f(x))/t| \leq n$ for all t, so $f \in E_n$.

Claim: E_n is nowhere dense (ND). Given $f \in E_n$ and $\epsilon > 0$, we need $g \in B(f, 3\epsilon)$ with $g \notin E_n$. By uniform continuity of f, we can find $\delta > 0$ s.t. $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$. Take the following function. to be g.



Then we are done by Baire.

3 Spaces of Continuous Functions

3.1 Existence of Continuous Functions

Definition 3.1. A Hausdorff space X is normal if any two disjoint closed sets can be separated by two disjoint open sets.

Proposition 3.2. *Compact Hausdorff $\implies$ Normal.*

Proof. Part IB Analysis and Topology ES3 Q8. □

Theorem 3.3 (Urysohn's lemma). *Let X be normal, $A, B \subseteq X$ be disjoint closed sets. Then $\exists$ a cts function $f : X \to [0,1]$ s.t. $f \equiv 0$ on A and $f \equiv 1$ on B.*

Remark 10.

- Usually use normality in the following equivalent form. For $A \subseteq V$ where A is closed and V is open, there exists an open U with $A \subseteq U$ and $\bar{U} \subseteq V$.
- If X is Hausdorff and has the property in Urysohn lemma, then X is normal. Take $f^{-1}(\{1/4\})$ and $f^{-1}(\{3/4\})$, so this characterizes normal spaces.
- This is trivial for metric spaces. Take $f(x) = \frac{d(x,A)}{d(x,A)+d(x,B)}$.
- If K is compact Hausdorff (and infinite), then $C(K)$ is infinite dimensional. Pick an infinite sequence (x_n) with distinct terms (possible as the set is infinite). For $n \geq 2$, pick $f_n \in C(K)$ with $f_n = 0$ on $\{x_1, ..., x_n\}$ and $f_n = 1$ on $\{x_{n+1}\}$. It's easy to see that $\{f_n\}$ is an L.I. set.

Proof. Let $A, B \subseteq X$ be disjoint closed sets. Let $V = B^c$. Find $O_{1/2}$ open s.t. $A \subseteq O_{1/2}$ and $\bar{O}_{1/2} \subseteq V$. Continue, find $O_{1/4}, O_{3/4}$ with $A \subseteq O_{1/4}$, $\bar{O}_{1/4} \subseteq O_{1/2}$ and $\bar{O}_{1/2} \subseteq O_{3/4}$, $\bar{O}(3/4) \subseteq V$. Continue, we get O_q for all dyadic rational $q \in (0,1]$. The family $\{Q_q\}$ has the properties that

- $A \subseteq O_q \subseteq V$ for all q;
- $\bar{O}_q \subseteq O_r$ if $q < r$;

- Define $O_1 = X$.

Define $f(x) = \inf\{q : x \in O_q\}$. To show continuity, it is enough to consider the preimages of $(-\infty, a)$ and $(-\infty, a)$. for $a \in (0,1)$.

- $f(x) < a \Leftrightarrow x \in O_q$ for some $q < a$, $\Leftrightarrow x \in \bigcup_{q<a} O_q$ which is open;
- $f(x) > a \Leftrightarrow x \notin O_q$ for some $q > a$, i.e., $x \notin \bar{O}_r$ for some $r > a \Leftrightarrow x \in \bigcup_{r>a} \bar{O}_r^c$ open.

□

Theorem 3.4 (Tietze extension theorem)**.** *X normal, $Y \subseteq X$ closed, $f : Y \to \mathbb{R}$ bounded cts. Then f extends to a cts function $g : X \to \mathbb{R}$ with $\|g\|_\infty = \|f\|_\infty$.*

Remark 11.

- The condition $\|g\| = \|f\|$ is trivial. If $g : X \to \mathbb{R}$ extends to f with $\|f\| = 1$, then the map

$$g'(x) = \begin{cases} g(x) & g(x) \in [-1,1] \\ 1 & g(x) > 1 \\ -1 & g(x) < -1 \end{cases}$$

  also extends f ctsly.
- Y being closed is necessary, e.g., $\sin(1/x)$ on $(0,1] \subseteq [0,1]$.
- Tietze extends the result of Urysohn lemma.

Proof. WLOG, $\|f\| = 1$. Let $A = \{x \in Y : f(x) < -1/3\}$, $B = \{x \in Y : f(x) > 1/3\}$. By Urysohn, there exists a cts $g_1 : X \to [-1/3, 1/3]$ with $g_1 = -1/3$ on A and $g_1 = 1/3$ on B. Note that $|f(x) - g_1(x)| \le 2/3$ for all $x \in Y$, so $\|f - g_1\|_\infty \le 2/3$. Repeat the process with $(f - g)$ in place of f to obtain $g_2 : X \to [-2/9, 2/9]$ s.t. $\|f - g_1 - g_2\| \le (2/3)^2$. We thus get a sequence of cts functiosn $g_1, g_2, ... : X \to \mathbb{R}$, s.t. $\|g_n\| \le \frac{1}{3}(\frac{2}{3})^{n-1}$ and $\|f - \sum_{i=1}^n g_i\|_\infty \le (\frac{2}{3})^n \to 0$ as $n \to \infty$. Let $g = \sum_i g_i$ which converges (being uniformly Cauchy). Have $\|g\|_\infty \le 1$ (can bound this by a geometric series), and $\|f - g|_Y\| = 0$, so $f = g|_Y$. □

Proof. One can also proceed slightly differently by applying Riesz's lemma and open mapping lemma. The image of the restriction map is dense, and each function in the image of some function of the same sup norm. Open mapping lemma then implies that the restriction map is surjective. □

3.2 Compactness in $C(K)$

Definition 3.5. For K compact Hausdorff and $S \subseteq C(K)$, S is equicontinuous at x if

$$\forall \epsilon > 0,\ \exists \text{ nbd } N \ni x \text{ s.t. } \forall f \in S,\ y \in N \implies |f(x) - f(y) < \epsilon$$

We say that S is equicontinuous if it is equicontinuous at x for every $x \in K$.

Example 3.6. $\{\sin(n + x) : n \in \mathbb{N}\}$ is equicontinuous since $|x - x'| < \epsilon \implies |f(x) - f(x')| < \epsilon$ (MVT + all derivatives ≤ 1). But $\{\sin(nx) : n \in \mathbb{N}\}$ is not equicontinuous at 0.

Theorem 3.7 (Arzela-Ascoli)**.** *Let K be compact Hausdorff and $S \subseteq C(K)$. Then S is compact if and only if S is closed, bounded and equicontinuous.*

Proof. '$\Rightarrow$': If S is compact, then it is closed and totally bounded (hence bounded) by IB analysis and topology. Let $\epsilon > 0$ be given. Let $x \in K$. Since S is totally bounded, we can find $f_1, ..., f_n \in S$ s.t. $S \subseteq \bigcup_{i=1}^n B(f_i, \epsilon)$. For each i, f_i is continuous at x, so there exists open U_i with $x \in U_i$ s.t. $\forall y \in U_i$, $|f_i(y) - f_i(x)| < \epsilon$. Let $U = \bigcap_{i=1}^n U_i$. Then U is an open nbd of x and $\forall y \in U$ and $\forall i$, $|f_i(y) - f_i(x)| < \epsilon$. But then for any $f \in S$, we have $\|f - f_i\|_\infty < \epsilon$ for some i. Therefore

$$\forall y \in U,\ |f(y) - f(x)| \le |f(y) - f_i(y)| + |f_i(y) - f_i(x)| + |f_i(x) - f(x)| < 3\epsilon$$

'$\Leftarrow$:': Conversely S is closed, so sufficient to prove total boundedness. Given $\epsilon > 0$, we know that S is equicontinuous, so $\forall x \in K$, there exists open $U_x \ni x$ s.t. $\forall f \in S$, $\forall y \in U_x$: $|f(y) - f(x)| < \epsilon$.

Then $(U_x)_{x\in K}$ is an open cover of K. By compactness of K, we can reduce to a fintie subcover, say $K = U_{x_1} \cup ... \cup U_{x_n}$. For $f \in S$ define $\hat{f} \in \mathbb{R}^n$ by $\hat{f} = (f(x_1), ..., f(x_n))$. $\hat{f}$ contains all the information of f up to ϵ. Let $\hat{S} = \{\hat{f} : f \in S\} \subseteq \mathbb{R}^n$ equipped with the uniform norm $\|\cdot\|_\infty$. Then $\hat{S}$ is bounded as S is bounded. Hence $\hat{S}$ is totally bounded as we are in a finite dimensional space. Hence $\hat{S} \subseteq B(\hat{f}_1, \epsilon) \cup ... \cup B(\hat{f}_k, \epsilon)$ for some $f_1, ..., f_k \in S$. Then $\forall f \in S$, we have $\hat{f} \in B(\hat{f}_r, \epsilon)$ for some r, i.e., $|f(x_i) - f_r(x_i)| < \epsilon$ for all i. So $\forall x \in K$, we have $x \in U_{x_i}$ for some i so

$$|f(x) - f_r(x)| \le |f(x) - f(x_i)| + |f(x_i) - f_r(x_i)| + |f_r(x_i) - f_r(x)| < 3\epsilon$$

This means that $f \in B(f_1, 3\epsilon) \cup ... \cup B(f_k, 3\epsilon)$, i.e., S is totally bounded. □

Remark 12. (i) Identical proof works for $C_{\mathbb{C}}(K)$.

(ii) Above proof shows that S totally bounded iff S bounded and equicontinuous.

Arzéla-Ascoli s often useful in showing that operators to $C(K)$ are compact.

Example 3.8 (Integral Operators)**.** Let $g \in C([0,1]^2)$ be fixed. For $f \in C[0,1]$ define $T(f) \in C[0,1]$ by

$$T(f)(x) = \int_0^1 g(x,t)f(t)dt$$

for $x \in [0,1]$. (e.g. $T(f)(x) = \int_0^1 e^{-xt}f(t)dt$) $T : C[0,1] \to C[0,1]$ is the integral operator with kernel g. We observe that T is linear and continuous since $|Tf(x)| \le \|g\|_\infty \|f\|_\infty$.

Claim: T is compact. Proof of claim: Need to show image of unit ball is equicontinuous. Given $\epsilon > 0$, there exists $\delta > 0$ s.t.

$$|x - x'| < \delta,\ t \in [0,1] \implies |g(x,t) - g(x',t)| < \epsilon$$

(This is uniform convergence of g) So for $f \in C[0,1]$ with $\|f\|_\infty = 1$, we have

$$|x - x'| < \delta \implies |Tf(x) - Tf(x')| \le \int_0^1 |g(x,t) - g(x',t)||f(t)|dt \le \epsilon$$

3.3 Density of $C(K)$

Let K be compact Hausdorff.

Definition 3.9. A subalgebra of $C(K)$ is a subspace such that $f, g \in A \implies fg \in A$.

Example 3.10. (i) $\{f \in C[0,1] : \text{poly}\}$ (dense)

Definition 3.11. For $f, g \in C_{\mathbb{R}}(K)$ define $f \vee g$ 'f max/meet g' by $(f \vee g)(x) = \max(f(x), g(x))$ and $f \wedge g$ 'f min/join g' by $(f \wedge g)(x) = \min(f(x), g(x))$.

Note that $\wedge$ and $\vee$ acts on $C_{\mathbb{R}}(K)$. (They are called lattice operations.)

Definition 3.12. A sublattice of $C_{\mathbb{R}}(K)$ is a subset A s.t. $f, g \in A \implies f \vee g, f \wedge g \in A$.

Example 3.13. (i) $\{f \in C_{\mathbb{R}}(K) : \forall x, f(x) \ge 0\}$

(ii) $\{f \in C_{\mathbb{R}}(K) : \|f\|_\infty \le 1\}$

(iii) $\{f \in C_{\mathbb{R}}[0,1] : f \text{ poly}\}$ is not a sublattice.

Note that the concept of sublattice makes sense for $C_{\mathbb{R}}(K)$ but not for $C_{\mathbb{C}}(K)$.

Lemma 3.14. *K compact Hausdorff, $A \subseteq C_{\mathbb{R}}(K)$ sublattice. Suppose A approximates f at every pair of pts (i.e., $\forall \epsilon > 0$, $\forall x, y \in K$, $\exists g \in A$ s.t. $|g(x) - f(x)|, |g(y) - f(y)| < \epsilon$). Then A approximates f uniformly.*

Proof. Let $\epsilon > 0$. For each $x, y \in K$, find $g_{xy} \in A$ s.t. $|g_{xy}(x) - f(x)|, |g_{xy}(y) - f(y)| < \epsilon$. Define $V_{xy} = \{z \in K : |g_{xy}(z) - f(z)| < \epsilon\}$. Then V_{xy} is an open nbd of x and y. Fix $x \in K$, we have an open cover $\{V_{xy} : y \in K\}$. Pass to a finite subcover $V_{xy_1}, ..., V_{xy_n}$. Put $g_x = g_{xy_1} \wedge \cdots \wedge g_{xy_n}$. Then g_x satisfies the property $\forall y \in K$, $g_x(y) < f(x) + \epsilon$. Define $U_x = \{z : |g_x(z) - f(z)| < \epsilon\}$. Pass to a finite cover $K = U_{x_1} \cup \cdots \cup U_{x_m}$. Put $g = g_{x_1} \vee \cdots \vee g_{x_m}$. Then $\forall y \in K$, have $g(x) > f(x) - \epsilon$. Therefore $|f(x) - g(x)| < \epsilon$ for all $x \in K$. □

Aim to prove that closed subalgebra are always sublattices.

Lemma 3.15. $x \mapsto |x|$ on $[-1,1]$ can be unif. approximated by poly.

Proof. Enough to approximate $\sqrt{x^2+\epsilon}$ for $\epsilon > 0$ because $|\sqrt{x^2+\epsilon} - |x|| = |\frac{\epsilon}{\sqrt{x^2+\epsilon}+|x|}| \leq \sqrt{\epsilon}$. Use the holo'c function $(z+\epsilon)^{1/2}$ on $\Re(z) > -\epsilon$ to obtain a uniformly convergent series expansion on $[0,1]$, e.g., Taylor expand at $1/2$. (It is locally uniformly convergent on the disk of radius $1/2+\epsilon$, so restrict to the closed disk of radius $1/2$ it is unif. convergent.) Now, the same truncated series evaluated at x^2 is unif. convergent on $[-1,1]$ □

Corollary 3.16. *K is compact Hausdorff, $A \subseteq C(K)$ closed subalgebra. Then A is a sublattice.*

Proof. Enough to show that $f \in A \implies |f| \in A$ because $f \vee g$ and $f \wedge g$ can be expressed in terms of f, g and $|f-g|$. WLOG, assume $\|f\|_\infty \leq 1$. Given $\epsilon > 0$, find $P(t)$ poly s.t. $|P(t) - |t|| < \epsilon$ on $[-1,1]$. $|P(f) - |f(t)|| < \epsilon$ since $\|f\|_\infty \leq 1$. □

Theorem 3.17 (Stone-Weierstrass)**.** *K compact Hausdorff. $A \subseteq C(K)$ subalgebra s.t.*

(i) A contains the constants

(ii) A separates points of K, i.e., $\forall x, y \in K$ $x \neq y$, $\exists f \in A$ s.t. $f(x) \neq f(y)$.

Then A is dense in $C(K)$.

Proof. $\bar{A}$ closed subalgebra, so a sublattice. Given $x, y \in K$ and $\epsilon > 0$, can find $g \in A$ s.t. $g(x) = f(x)$ and $g(y) = f(y)$. The previous lemma implies that $\bar{A}$ approximates f unif. so $\bar{A}$ is dense in $C(K)$, so $\bar{A} = C(K)$. □

Remark 13.

4 Hilbert Spaces

Let X be a real or complex vector space.

Definition 4.1. An inner product on X is a function $(-,-) : X \times X \to \mathbb{F}$ (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) s.t. for all $x, y, z \in X$ and $\lambda, \mu \in \mathbb{F}$,

(i) $(\lambda x + \mu y, z) = \lambda(x,z) + \mu(y,z)$

(ii) $(y,x) = \overline{(x,y)}$

(iii) $(x,x) \geq 0$, with $(x,x) = 0$ iff $x = 0$.

Examples include the usual inner product on ℓ^2 and L^2. Note that an inner product induces a norm by $\|x\| = (x,x)^{1/2}$.

Theorem 4.2. *Let X be an inner product space. Then*

(i) (Cauchy-Schwarz) $\forall x, y \in X$, $|(x,y)| \leq \|x\|\|y\|$

(ii) (Triangle inequality) $\forall x, y \in X$, $\|x+y\| \leq \|x\| + \|y\|$

Proof. (i): WLOG, assume $(x,y) \in \mathbb{R}$. Then $\forall y \neq 0$ and $t \in \mathbb{R}$, have $\|x+ty\|^2 = (x+ty, x+ty) = \|x\|^2 + t^2\|y\|^2 + 2t(x,y) \geq 0$. Hence $4(x,y)^2 - 4\|x\|^2\|y\|^2 \leq 0$. Rearrange.

(ii) $\|x+y\|^2 = (x+y, x+y) = \|x\|^2 + \|y\|^2 + 2\Re(x,y) \leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2$ □

Definition 4.3. A Hilbert space is a complete inner product space.

Example 4.4. ℓ^2 is Hilbert, but $C[0,1]$ with $(f,g) = \int_0^1 f\bar{g}$ is not Hilbert.

Proposition 4.5 (Polarization identity)**.** *Let X be an inner product space; $x, y \in X$. Then*

Real Case: $(x,y) = \frac{1}{2}(\|x+y\|^2 - \|x\|^2 - \|y\|^2)$

Complex Case: $(x,y) = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2)$

Proof. Expand $\|x+y\|^2$ in each case. □

Proposition 4.6 (Parallelogram Law). *Let X be an inner product space; $x, y \in X$. Then* $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

Proof. Compute. □

Remark 14. The parallelogram law characterizes inner product spaces in the sense that if a normed space X satisfies the conclusion of the parallelogram law, then the polarization identity defines a valid inner product on X which induces its norm. In particular, a normed space X is an inner product space if and only if every 2-dimensional subspace of X is an inner product space.

Definition 4.7. X inner product space. $x, y \in X$ are said to be orthogonal, written $x \perp y$ if $(x, y) = 0$.

Proposition 4.8 (Pythagoras). *X an inner product space; $x, y \in X$. Then* $x \perp y \implies \|x+y\|^2 = \|x\|^2 + \|y\|^2$

Proof. Compute. □

Similarly, if $x_1, ..., x_n$ are pairwise orthogonal, then $\|\sum_i x_i\|^2 = \sum_i \|x_i\|^2$.

Theorem 4.9. *Let X be an inner product space. Then its completion $\tilde{X}$ is also an inner product space.*

Proof. For $x, y \in \tilde{X}$, choose (x_n) and (y_n) in X s.t. $x_n \to x$ and $y_n \to y$. Define $(x, y) = \lim_{n\to\infty}(x_n, y_n)$. Easy to check various properties and that the induced norm is complete. □

Example 4.10. L^2 is the completion of $C[0, 1]$ with respect to $\|\cdot\|_2$.

Recall that the distance from a point to a closed set needs not be attained even in ℓ^2.

Theorem 4.11 (Closest Point Theorem). *Let H be a Hilbert space, S a closed subspace of H, $x \in H$. Then there exists a unique $y \in S$ s.t. $\|x - y\| = d(x, S)$.*

Proof. Let $d = d(x, S)$. Choose (y_n) in S s.t. $\|x - y_n\| \to d$. We now use the parallelogram law to prove that y_n is Cauchy. The key ingredient is the parallelogram law. We have

$$2\|x - y_n\|^2 + 2\|x - y_m\|^2 = \|2x - y_n - y_m\|^2 + \|y_n - y_m\|^2$$

Rearrange,

$$\begin{aligned}\|y_n - y_m\|^2 &= 2\|x - y_n\|^2 + 2\|x - y_m\|^2 - 4\|x - \frac{1}{2}(y_n + y_m) \\ &\le 2\|x - y_n\|^2 + 2\|x - y_m\| - 4d^2\end{aligned}$$

By completeness and closedness, this sequence has a limit $y \in S$ s.t. $d(x, y) = d$.

To see uniqueness, note that if z also satisfies the above, then $\|y - z\|^2 = 2\|x - y\|^2 + 2\|x - z\|^2 - 4\|x - \frac{1}{2}(y + z)\|^2 \le 2d^2 + 2d^2 - 4d^2 = 0$, so $y = z$. □

Remark 15. The same proof shows that the distance to a closed convex set is attained.

If X is an inner product space, $x \in X$, then we define $x^\perp = \{y \in X : (x, y) = 0\}$, which is a closed subspace. For $S \subseteq X$, $S^\perp = \{y \in X : \forall x \in S,\ (x, y) = 0\} = \bigcap_{x \in S} x^\perp$, and $S^\perp$ is a closed subspace of X. It is clear that $S \subseteq S' \implies S'^\perp \subseteq S^\perp$.

Theorem 4.12. *Let H be a Hilbert space, F a closed subspace of H. Then $H = F \oplus F^\perp$, i.e., F has an orthogonal complement.*

Proof. It is clear that $F \cap F^\perp = \{0\}$. WTS $F + F^\perp = H$. Let $x \in H$. Choose $y \in F$ s.t. $\|x - y\| = d(x, F)$. We claim that $x - y \in F^\perp$. If not, then pick $z \in F$ s.t. $(x - y, z) \neq 0$. WLOG, assume $(x - y, z)$ is real and > 0. Let $t \in \mathbb{R}^+$. We have $\|x - y\|^2 \le \|x - (y + tz)\|^2 \le \|x - y\|^2 + t^2\|z\|^2 - 2t(x - y, z)$. So: $2t(x - y, z) - t^2\|z\|^2 \le 0$. False for sufficiently small t. □

Note that $S \subseteq X \implies S \subseteq (S^\perp)^\perp$.

Corollary 4.13. *H a Hilbert space. Then*

(i) F *a closed subspace of* $H \implies (F^\perp)^\perp = F$

(ii) $S \subseteq H \implies (S^\perp)^\perp = \overline{\langle S \rangle}$

(iii) $S \subseteq H$ has dense linear span $\Leftrightarrow S^\perp = \{0\}$

Proof. Just check. □

If H is a Hilbert space, then for each $y \in H$ one may define $\theta_y : H \to \mathbb{C}$ by $\theta_y(x) = (x, y)$, which is bounded linear by Cauchy-Schwarz. We also deduce that $\|\theta_y\| \leq \|y\|$. But note that $\theta_y(y) = \|y\|^2$, so $\|\theta_y\| = \|y\|$.

Theorem 4.14 (Riesz Representation Theorem). *H a Hilbert space, $f \in H^*$. Then $\exists y \in H$ s.t. $f = \theta_y$.*

Proof. WLOG $f \neq 0$. Put $E = \ker f$. Then E is a closed subspace of H and $E \neq H$, so $E^\perp \neq \{0\}$. Also, $\dim E < 2$ because $x, y \in H \implies f(\lambda x + \mu y) = 0$ for some $\lambda, \mu \neq 0$, so $E^\perp = \langle y \rangle$ for some $y \in H$. WLOG, assume $\|y\| = 1$. Define $u = \overline{f(y)}y$. Then $f(u) = \|u\|^2$. Now, for any $x \in H$, write $x = z + \lambda u$ for some $z \in E$, $\lambda \in \mathbb{C}$. Then, $(x, y) = (z = \lambda y, y) = \lambda \|y\|^2 = f(x)$. Hence, we are done. □

Corollary 4.15. *H Hilbert. The map $\Theta : H \to H^*$, $y \mapsto \theta_y$ is an isometric, conjugate-linear isomorphism, i.e., H is self-dual.*

Proof. Have $\|\theta_y\| = \|y\| \; \forall y \in H$, so Θ is an isometry. It's surjective by Riesz Representation Theorem. Clearly conj. linear by definition. □

4.1 Orthonormal Bases

Definition 4.16. A sequence (x_n) in an inner product space X is an orthonormal sequence if it consists of pairwise orthogonal unit vectors. It is an orthonormal basis if $\langle x_n : n \in \mathbb{N} \rangle$ has dense linear span.

We use the same terminology for finite sequences.

Theorem 4.17 (Gram-Schmidt Process). *Let (x_n) be a linearly independent sequence in an inner product space X. Then there exists an orthonormal sequence (e_n) s.t. $\langle e_1, ..., e_n \rangle = \langle x_1, ..., x_n \rangle$ for all n.*

Proof. Let $e_1 = x_1 / \|x_1\|$. Then let $e_2' = x_2 - (x_2, e_1)e_1$ and define $e_2 = e_2' / \|e_2'\|$. Continue inductively. □

Corollary 4.18. *Let X be a separable inner product space. Then X has an orthonormal basis*

Proof. Let (x_n) be a sequence which is dense in X. WLOG, assume (x_n) is linearly independent by removing any x_n depending on its predecessors. Apply Gram-Schmidt. □

Example 4.19. The normalized Legendre polynomials are obtained by performing Gram-Schmidt on the dense sequence $1, t^2, t^3, ...$ in $X = C_\mathbb{R}[-1, 1]$ with the 2-norm. In fact, the nth one is a multiple of $\frac{d^n}{dt^n}((t^2 - 1)^n)$.

Corollary 4.20. *Let X be an n-dimensional inner product space. Then X is isometrically isomorphic to ℓ_2^n.*

Proof. Let $e_1, ..., e_n$ be an orthonormal basis of X. Define $T : X \to \ell_2^n$ by $T(\sum_i \lambda_i e_i) = (\lambda_1, ..., \lambda_n)$. T is linear and bijective. Also, $\|(\lambda_1, ..., \lambda_n)\|_2^2 = \sum_i |\lambda_i|^2 = \|\sum_i \lambda_i e_i\|^2$, i.e., T is norm-preserving. □

Remark 16. Isometric isomorphism also preserves the inner product by polarization.

We now aim to show that every separable infinite-dimensional Hilbert space is isometrically isomorphic to ℓ_2.

Proposition 4.21. *H Hilbert space; (e_n) an orthonormal sequence in H. Then for any $\lambda_1, \lambda_2, ..., \in \mathbb{C}$,*

$$\sum_n \lambda_n e_n \text{ converges } \Leftrightarrow (\lambda_n) \in \ell_2$$

Proof. '$\Rightarrow$': $\|\sum_{n=1}^N \lambda_n e_n\| \to \|\sum_{n=1}^\infty \lambda_n e_n\|$ as $N \to \infty$. But $\|\sum_{n=1}^N \lambda_n e_n\| = \sum_{n=1}^N |\lambda_n|^2$, so $\sum_{n=1}^\infty |\lambda_n|^2 < \infty$.

'$\Leftarrow$': $\|\sum_{n=N}^M \lambda_n e_n\|^2 = \sum_{n=N}^M |\lambda_n|^2 \to 0$ as $M, N \to \infty$, so $\sum_{n=1}^N \lambda_n e_n$ is Cauchy. □

Corollary 4.22 (Riesz-Fisher Theorem). *H Hilbert; (e_n) an orthonormal sequence in H. Then for all $c \in \ell_2$, there exists $x \in H$ s.t. $(x, e_n) = c_n$ for all n.*

Proof. Let $x = \sum_n c_n e_n$ which converges as $c \in \ell_2$. Then $(\sum_{n=1}^N c_n e_n, e_k) = c_k$ for all $N \geq k$, so $(x, e_k) = c_k$ for all k. □

Definition 4.23. For an orthonormal sequence (e_n) in an inner product space X, we call (x, e_k) the kth (Fourier) coefficient of x with respect to (e_n).

We will show that for an orthonormal basis, $x = \sum_n (x, e_n) e_n$ for all x.

Remark 17. This is false in a general Banach space. For instance, $1, t, t^2, \ldots$, have dense linear span in $C_{\mathbb{R}}[-1, 1]$ but $|t| \neq \sum c_k t^k$.

Theorem 4.24 (Bessel's inequality)**.** *H Hilbert; (e_n) an orthonormal sequence. Then for all $x \in H$, $\sum_{n=1}^\infty |(x, e_n)|^2 \leq \|x\|^2$. In particular, $((x, e_n))_{n=1}^\infty \in \ell_2$.*

Proof. Put $y = \sum_{n=1}^k (x, e_n) e_n$. We have $(y, e_n) = (x, e_n)$ for all $n \leq k$, so $(x - y) \perp e_n$ for all $n \leq k$, so $(x - y) \perp y$, so By Pythagoras, $\|x\|^2 = \|x - y\|^2 + \|y\|^2 \geq \|y\|^2$, and we are done. □

Theorem 4.25. *H Hilbert; (e_n) an orthonormal basis in H. Then for all $x \in H$, $x = \sum_{n=1}^\infty (x, e_n) e_n$.*

Proof. $((x, e_n))_{n=1}^\infty \in \ell_2$ by Bessel, so $\sum_n (x, e_n) e_n$ converges to some $y \in H$. Note that $(y, e_n) = (x, e_n)$ for all n, so $(y - x) \perp e_n$ for all n, so $y - x \in \overline{\langle e_n : n \in \mathbb{N} \rangle}^\perp = \{0\}$, so $y = x$. □

Remark 18. The previous two theorems also hold for inner product spaces: if (e_n) is an orthonormal basis in X, then (e_n) is an orthonormal basis in its completion $\tilde{X}$ as X is dense in $\tilde{X}$.

Example 4.26. $C_{\mathbb{C}}[0, 2\pi]$ with the 2-norm. Then $(e^{in\theta})_{n \in \mathbb{Z}}$ is an orthonormal basis. So we get Fourier series which converges in $\| \cdot \|_2$. The same holds for any $f \in L^2[0, 2\pi]$.

For $f \in C_{\mathbb{C}}[0, 2\pi]$, $f(0) = f(2\pi)$. Write

$$S_k = \sum_{n=-k}^{k} (f, e^{in\theta}) e^{-in\theta}$$

The above says $S_k \to f$ in $\| \cdot \|_2$. Need not have uniform convergence or even pointwise convergence. Fejer's Theorem says that

$$\frac{S_1 + \cdots + S_k}{k} \to f$$

uniformly.

Corollary 4.27 (Parseval)**.** *H Hilbert, (e_n) orthonormal basis for H. Then*

(i) For all $x \in H$, $\|x\|^2 = \sum_n |(x, e_n)|^2$

(ii) For all $x, y \in H$, $(x, y) = \sum_n (x, e_n) \overline{(y, e_n)}$.

Proof. (i): $\| \sum_{n=1}^k (x, e_n) e_n \|^2 = \sum_{n=1}^k |(x, e_n)|^2$. Let $k \to \infty$, $\|x\|^2 = \sum_n |(x, e_n)|^2$.

(ii): $(\sum_{n=1}^k (x, e_n) e_n, \sum_{n=1}^k (y, e_n) e_n) = \sum_{n=1}^k (x, e_n) \overline{(y, e_n)}$. Let $k \to \infty$. □

Corollary 4.28. *Let H be a separable infinite dimensional Hilbert space. Then H is isometrically isomorphic to ℓ_2.*

Proof. Choose an orthonormal basis (e_n) for H. Define $T : H \to \ell_2$ by $T(x) = ((x, e_n))_{n=1}^\infty$. This is well-defined by Bessel. T is clearly linearl and norm preserving. It is surjective by Riesz-Fischer. □

4.2 Matrices of Linear Operators

Definition 4.29. Let H be a Hilbert space with an orthonormal basis (e_n). For $T \in L(H)$, the matrix of T w.r.t. (e_n) is $A = (a_{ij})_{i,j=1}^\infty$ where $a_{ij} = (Te_j, e_i)$.

Thus, $Te_j = \sum_{i=1}^\infty a_{ij} e_i$, so the matrix of T determines T.

Remark 19. Not every matrix comes from a $T \in L(H)$.

4.3 Adjoints

Theorem 4.30. *H Hilbert, $T \in L(H)$. Then there exists a unique map T^* s.t. for all $x, y \in H$, $(Tx, y) = (x, T^*y)$. Moreover, $T^* \in L(H)$.*

Remark 20. If T has a matrix A with respect to orthonormal basis, then T^* has matrix $A^\dagger$ with respect to this basis. However, we cannot define T^* this way because

1) Might depend on the orthonormal basis

2) H might not have an orthonormal basis (e.g. if it's not separable)

3) $A^\dagger$ might not be the matrix of a continuous linear map

Definition 4.31. T^* is the adjoint of T.

Proof. For each $y \in H$, $x \mapsto (Tx, y)$ is a bounded linear functional. By Riesz representation, there exists $z \in H$ s.t. $\theta_z = (x \mapsto (Tx, y))$ so that $(Tx, y) = (x, z)$. Define $T^*y = z$.

Uniqueness: note that $(Tx, y) = (x, z)$ for all x. If z' is another candidate, then $(x, z) = (x, z')$ for all x, so $z = z'$.

Linearity: just compute.

Boundedness: For fixed $y \in H$, $|(x, T^*y)| = |(Tx, y)| \leq \|T\|\|x\|\|y\|$ for all $x \in H$, so $|(x, T^*y)| \leq \|T\|\|y\|$ for all $x \in H$ with $\|x\| \leq 1$, so $\|T^*y\| \leq \|T\|\|y\|$ (using that $\|z\| > k \implies (\frac{z}{\|z\|}, z) > k$). So $T^* \in L(H)$ with $\|T^*\| \leq \|T\|$. □

Example 4.32. $H = \ell_2$. The adjoint of the left shift operator is the right shift operator.

Proposition 4.33. *H Hilbert, $S, T \in L(H)$. Then*

(i) $(\lambda S + \mu T)^* = \bar{\lambda} S^* + \bar{\mu} T^*$ *for all* $\lambda, \mu \in \mathbb{C}$.

(ii) $(ST)^* = T^* S^*$

(iii) $(T^*)^* = T$

(iv) $\|T^*\| = \|T\|$

(v) $\|T^*T\| = \|T\|^2$

Proof. (i), (ii), and (iii) are easy to prove. (iv) follows from (iii) since the reversed inequality $\|T\| \leq \|T^*\|$ also holds. For (v), certainly $\|T^*T\| \leq \|T\|^2$. Also, $\|Tx\|^2 = (x, T^*Tx) \leq \|x\|^2\|T^*T\|$, so $\|T\|^2 \leq \|T^*T\|$. □

Remark 21. It's not true in general that $\|T^2\| = \|T\|^2$, e.g., consider a nilpotent operator.

Definition 4.34. H Hilbert; $T \in L(H)$. Then T is Hermitian (or self-adjoint) if $T^* = T$.

Example 4.35. Orthogonal projections onto a closed subspace is Hermitian.

Proposition 4.36. *H complex Hilbert space; $T \in L(H)$. Then there exists T_1, T_2 Hermitian with $T = T_1 + iT_2$*

Proof. $T = \frac{1}{2}(T + T^*) + i \cdot \frac{i}{2}(T^* - T)$. □

Remark 22. This decomposition is unique. If $T_1 + iT_2 = T_1' + iT_2'$, then taking adjoint and manipulate a little bit, you get $T_1 - T_1' = 0$ and $T_2 - T_2' = 0$.

5 Spectral Theory

Let X be a Banach space, $T \in L(X)$. We know that $T \in L(X)$ is invertible if and only if T is a bijection and T^{-1} is continuous. This is equivalent to T being bijective by inversion theorem. So: $T \in L(X)$ is invertible if and only if T is injective and surjective. Note that both conditions are necessary (e.g., left shift and right shift).

Theorem 5.1. *X Banach space, $T \in L(X)$. Then $\|T\| < 1$ implies that $I - T$ is invertible with $(I-T)^{-1} = \sum_{n=0}^{\infty} T^n$*

Proof. Note that $(I-T)\sum_{n=0}^{k} T^n = I - T^{k+1}$. Let $k \to \infty$, this expression converges to I in operator norm. Similar argument holds for the other composition. □

For a Banach space X, we write $GL(X)$ for the set of invertible bounded linear operators $X \to X$.

Theorem 5.2. *X Banach, then*

(i) G is open in $L(X)$

(ii) The function $T \to T^{-1}$ from G to itself is continuous,

(iii) Let (T_n) in G, $T \in L(X)$ with $T_n \to T$ but $T \notin G$, then $\|T_n^{-1}\| \to \infty$.

Proof. (i): For $T \in G$, have $T - S = T(I - T^{-1}S)$ for all $S \in L(X)$. Choose S with sufficiently small operator norm, in particular, $\|S\| < \frac{1}{\|T^{-1}\|}$, then $I - T^{-1}S$ is invertible. This shows that $B(T, \frac{1}{\|T^{-1}\|}) \subseteq G$.

(ii): For $\|S\| < \|T^{-1}\|^{-1}$, $(T-S)^{-1} - T^{-1} = \sum_{n=1}^{\infty}(T^{-1}S)^n T$ Taking operator norm, see that

$$\|(T-S)^{-1} - T^{-1}\| \leq \sum_{n=1}^{\infty} \|S\|^n \|T^{-1}\|^{n+1} \to 0$$

as $\|S\| \to 0$.

(iii): Given $\epsilon > 0$, for n sufficiently large s.t. $\|T_n - T\| < \epsilon$ we have $B(T_n, \frac{1}{\|T_n^{-1}\|}) \subseteq G$ and $T \notin G$ so $\|T_n^{-1}\|^{-1} < \epsilon$, so $\|T_n^{-1}\| > \frac{1}{\epsilon}$. □

Now we restrict out attention to complex Banach spaces.

Definition 5.3. Let X be a (complex) Banach space, $T \in L(X)$, $\lambda \in \mathbb{C}$. λ is an eigenvalue of T if there exists $x \in X$ non-zero with $Tx = \lambda x$. Such an x is called an eigenvector of T.

Example 5.4. $X = \ell_2$, $T =$ Right shift. Then 0 is not an eigenvalue of T, but T is not surjective and hence not invertible. In fact, T has no eigenvalues.

Definition 5.5. X complex Banach space with $T \in L(X)$. The spectrum of T is $\sigma(T) = \{\lambda \in \mathbb{C} : (T - \lambda I) \text{ not invertible}\}$.

Observe that for finite dimensional space, this is just the set of eigenvalues.

Remark 23. $\lambda \in \sigma(T)$ iff either $T - \lambda I$ not injective (λ is an e-value) or $T - \lambda I$ not surjective.

Proposition 5.6. *X a (complex) Banach space; $T \in L(X)$. Then $\sigma(T)$ is a closed subset of $\{z \in \mathbb{C} : |z| \leq \|T\|\}$. In particular, $\sigma(T)$ is compact.*

Proof. $\lambda \notin \sigma(T)$ iff $T - \lambda I \in G$. But G is open and $\lambda \mapsto T - \lambda I$ is a continuous function $\mathbb{C} \to L(X)$, so $\mathbb{C} \setminus \sigma(T)$ is open.

For $|\lambda| > \|T\|$, $T - \lambda I = -\lambda(I - \frac{I}{\lambda})$ which is invertible. □

Definition 5.7. $\mathbb{C} \setminus \sigma(T)$ is the resolvent set of T. The function $R : \mathbb{C} \setminus \sigma(T) \to L(X)$, $\lambda \mapsto (\lambda I - T)^{-1}$ is the resolvent function of T.

For $|\lambda| > \|T\|$, $R(\lambda) = \sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$.

Example 5.8. The spectrum of the left shift operator is the closed unit disk. Note that every value in the open unit disk is an eigenvalue (consider $(1, \lambda, \lambda^2, ...)$), and $\sigma(T)$ is a closed subset of $\{z \in \mathbb{C} : |z| \leq 1\}$.

Definition 5.9. X complex Banach space; $T \in L(X)$; $\lambda \in \mathbb{C}$. λ is an approximate eigenvalue of T if

$$\forall \epsilon > 0, \exists x \in X, \|x\| = 1 \text{ s.t. } \|Tx - \lambda x\| < \epsilon$$

Equivalently, λ is an approximate eigenvalue if there exists a sequence (x_n) in X with $\|x_n\| = 1$ for all n s.t. $\|Tx_n - \lambda x_n\| \to 0$. Such a sequence (x_n) is an approximate eigenvector of T.

Proposition 5.10. *X complex Banach; $T \in L(X)$; $\lambda \in \mathbb{C}$. Then*

(i) λ is an eigenvalue of $T \implies \lambda$ is an approximate eigenvalue of T.

(ii) λ is an approximate eigenvalue of $T \implies \lambda \in \sigma(T)$.

Proof. (i): Trivial.

(ii): If λ is an approx. e-value of T, then find sucha sequence (x_n). If $T - \lambda I$ is invertible, then $(T - \lambda I)^{-1}(Tx_n - \lambda x_n) \to 0$, so $x_n \to 0$. Contradiction. □

Example 5.11. 1. $Te_n = \frac{1}{2^n} e_n$, then 0 is an approx e-value but not an e-value.

2. T Right shift. 0 is not an approx e-value, but $0 \in \sigma(T)$.

Definition 5.12. Write $\sigma_{ap}(T)$ for the set of approximate eigenvalues. This is called the approximate point spectrum of T. Somtie write $\sigma_p(T)$ for the set of eigenvalues, called the point spectrum of T.

Theorem 5.13. *X a complex Banach space; $T \in L(X)$. Then $\partial\sigma(T) \subseteq \sigma_{ap}(T)$.*

Proof. For $\lambda \in \partial\sigma(T)$, there exists $\lambda_n \in \mathbb{C} \setminus \sigma(T)$ with $\lambda_n \to \lambda$. Then $T - \lambda_n I \to T - \lambda I$, so $\|(T - \lambda_n I)^{-1}\| \to \infty$ as $n \to \infty$. So there exists a sequence (x_n) in X s.t. $\|x_n\| \to 0$ and $\|(T - \lambda_n I)^{-1} x_n\| = 1$ for all n. Put $y_n = (T - \lambda_n I)^{-1} x_n$, then $\|y_n\| = 1$ for all n and $(T - \lambda_n I)y_n = x_n \to 0$, so $(T - \lambda I)y_n = 0$ (note: $(T - \lambda I)y_n = (T - \lambda_n I)y_n + (\lambda_n - \lambda)y_n$) □

Example 5.14. Let T be the right shift operator. Any value in the open unit disk is not an approximate e-value, so $\partial\sigma(T) \subseteq S^1$. Then either $D^1 \subseteq \sigma(T)$ or $D^1 \cap \sigma(T) = \varnothing$. Note that $0 \in \sigma(T)$, so the whole open unit disk must be in $\sigma(T)$, so $\sigma(T)$ is the closed unit disk.

Theorem 5.15 (Spectral Mapping Theorem)**.** *X complex Banach space; $T \in L(X)$. Let p be a non-constant complex polynomial. Then $\sigma(p(T)) = \{p(\lambda) : \lambda \in \sigma(T)\}$.*

Proof. For a fixed $\mu \in \mathbb{C}$, write $p(z) - \mu = c(z - \lambda_1) \cdots (z - \lambda_n)$, where $c, \lambda_1, ..., \lambda_n \in \mathbb{C}$ and $c \neq 0$. Then $p(T) - \mu I = c(T - \lambda_1 I) \cdots (T - \lambda_n I)$.

We claim that $p(T) - \mu I$ is invertible if and only if $T - \lambda_i I$ is invertible for all i.

Proof of claim. '$\Leftarrow$': Trivial

'$\Rightarrow$': If $T - \lambda_i I$ is not invertible, then $(T - \lambda_1 I) \cdots (T - \lambda_n I)$ is not invertible. Note that for all A, B, A not invertible implies that A not injective or A not surjective, which implies that BA or AB not invertible. Also note that $T - \lambda_j I$ commute. □

So $\mu \in \sigma(p(T))$ iff $T - \lambda_i$ not invertible for some i, but $\{\lambda_1, ..., \lambda_n\} = \{\lambda \in \mathbb{C} : p(\lambda) = \mu$. This shows that $\mu \in \sigma(p(T)) \Leftrightarrow \lambda \in \sigma(T)$ for some λ with $p(\lambda) = \mu$. □

Remark 24. We did not cheat here. It is not true that A not invertible implies AB not invertible, e.g., A left shift, B right shift.

Definition 5.16. X a complex Banach space; $T \in L(X)$. The spectral value of T is $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$.

Certainly, $r(T) \leq \|T\|$.

Example 5.17. $r(\text{Left shift}) = r(\text{Right shift}) = 1$.

Corollary 5.18. *X a complex Banach space, $T \in L(X)$. Then $r(T) \leq \inf_{n \geq 1} \|T^n\|^{1/n}$.*

Proof. $r(T^n) \leq \|T^n\|$, but $r(T^n) = r(T)^n$ by spectral mapping theorem, so $r(T) \leq \|T^n\|^{1/n}$. □

Example 5.19. For T nilpotent, then $r(T) = 0$, so we can have strict inequality $r(T) \leq \|T\|$.

Theorem 5.20 (Non-emptiness of the spectrum)**.** *H a (non-zero) Hilbert space, $T \in L(H)$. Then $\sigma(T) \neq \varnothing$.*

Remark 25. If $\dim H < \infty$, then we are done by FTA, which can be proved by Liouville's theorem. We will carry out a similar argument here.

Proof. Suppose $\sigma(T) = \varnothing$. Then the resolvent function R is defined on $\mathbb{C}$. Observe that

$$R(\lambda) - R(\mu) = (\lambda I - T)^{-1}((\mu I - T) - (\lambda I - T))(\mu I - T)^{-1}$$

so

$$\frac{R(\lambda) - R(\mu)}{\lambda - \mu} = -R(\lambda)R(\mu) \to -R(\lambda)^2$$

as $\mu \to \lambda$. For fixed $x, y \in H$, define $f : \mathbb{C} \to \mathbb{C}$ by $f(\lambda) = (R(\lambda)x, y)$. Then by direct computation, we see that f is entire.

Now, for $|\lambda| > \|T\|$, we have $\|(\lambda I - T)u\| \geq (|\lambda| - \|T\|)\|u\|$ for all $u \in H$, so $\|(\lambda I - T)^{-1}\| \leq \frac{1}{|\lambda| - \|T\|} \to 0$ as $|\lambda| \to \infty$. Thus $|f(\lambda)| \leq \|R(\lambda)\|\|x\|\|y\| \to 0$ as $|\lambda| \to \infty$. By Liouville, $f \equiv 0$. Thus $(R(\lambda)x, y) = 0$ for all λ, x, y, so $R(\lambda)x = 0$ for all λ, x. This is a massive contradiction. □

Remark 26. 1) If $T \in L(X)$, X Banach, we could look at $f(\lambda) = \phi(R(\lambda)x)$, where $x \in X$ and $\phi \in X^*$. We would get $f \equiv 0$, i.e., $\phi(R(\lambda)x)$ for all λ, x. By Hahn-Banach, $R(\lambda)x = 0$ for all λ, x, a contradiction.

2) Can rephrase the above proof in terms of $L(H)$-valued analytic functions. A function is called analytic if for all $\lambda \in \mathbb{C}$, there exists $S \in L(H)$ s.t. $\frac{g(\lambda)-g(\mu)}{\lambda-\mu} \to S$ as $\mu \to \lambda$. We proved that an $L(H)$-valued Liouville's theorem.

3) Have $R(\lambda) = \sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$ for $|\lambda| > \|T\|$. By considering $L(H)$-valued analytic functions one can show that this Laurent series converges for $|\lambda| > r(T)$. But one can also show that $\sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$ has radius of convergence $\limsup \|T^n\|^{1/n}$. Hence $r(T) \geq \limsup \|T^n\|^{1/n}$. But $r(T) \leq \inf_{n \geq 1} \|T^n\|^{1/n}$ by spectral mapping theorem, so $r(T) = \lim_{n \to \infty} \|T^n\|^{1/n}$. This is the spectral radius formula.

5.1 Spectral Theory of Hermitian Operators

In this section all spaces are over $\mathbb{C}$.

Proposition 5.21. *H Hilbert space; $T \in L(H)$. Then $\sigma(T^*) = \{\bar{\lambda} : \lambda \in \sigma(T)\}$.*

Proof. For any $T \in L(H)$, T invertible iff T^* invertible. Thus $T - \lambda I$ is invertible iff $T^* - \bar{\lambda} I$ is invertible. □

Example 5.22. We can deduce the spectrum of right shift using this result.

Theorem 5.23. *H Hilber space, $T \in L(H)$ Hermitian. Then $\sigma(T) \subseteq \mathbb{R}$.*

Proof. For λ an approximate eigenvalue of T, there exists (x_n) in H, $\|x_n\| = 1$ with $Tx_n - \lambda x_n \to 0$. Thus $(Tx_n - \lambda x_n, x_n \to 0)$, so $(Tx_n, x_n) \to \lambda$. But $\forall x \in H$, $(Tx, x) = (x, Tx)$, so $(Tx_n, x_n) \in \mathbb{R}$ for all n, so $\lambda \in \mathbb{R}$. We have $\partial\sigma(T) \subseteq \sigma_{ap}(T) \subseteq \mathbb{R}$, so $\sigma(T) \subseteq \mathbb{R}$. □

Corollary 5.24. *H Hilbert space, $T \in L(H)$ Hermitian. Suppose λ, μ are distinct evalues with e-vectors x, y, then $x \perp y$*

Proof. Just compute. □

Remark 27. For T Hermitian, $\sigma(T) = \partial\sigma(T)$ since it's a subset of $\mathbb{R}$.

Proposition 5.25. *H Hilbert, $T \in L(H)$ Hermitian. Then $r(T) = \|T\|$.*

Proof. WLOG, $\|T\| = 1$. We will show that 1 or -1 is an approximate eigenvalue. Choose a sequence (x_n) in X s.t. $\|x_n\| = 1$ for all n and $\|Tx_n\| \to 1$. Then

$$(Tx_n, Tx_n) = (T^2 x_n, x_n) \to 1$$

as $n \to \infty$. Now:

$$\begin{aligned} \|(T^2 - I)x_n\|^2 &= (T^2 x_n - x_n, T^2 x_n - x_n) \\ &= \|T^2 x_n\|^2 + 1 - 2\Re(T^2 x_n, x_n) \end{aligned}$$

Note that $\|T^2 x_n\| \leq \|T\|^2 \|x_n\| \leq 1$, so $\|(T^2 - I)x_n\|^2 \to 0$ as $n \to \infty$, i.e., $(T + I)(T - I)x_n \to 0$. If $(T - I)x_n \to 0$, then 1 is an approximate eigenvalue. If $(T - I)x_n \not\to 0$, then by passing to a subsequence if necessary, choose $\delta > 0$ s.t. $\|(T - I)x_n\| \geq \delta$ for all n. Then $(T + I)(T - I)x_n \to 0$ and thus -1 is an approxiamte eigenvalue. □

Remark 28. For T Hermitian, have $\|T^2\| = \|T\|^2$. Repeat to get $\|T^{2^n}\| = \|T\|^{2^n}$ for all n. The spectral radius formula gives $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n} = \|T\|$

Definition 5.26. X Banach, $T \in L(X)$, $Y \leq X$ subspace. We say that T acts on Y (or equivalently, Y is an invariant subspace for T) if $TY \subseteq Y$.

Examples include eigenspaces.

Remark 29 (Invariant subspace problem). Does every operator T on a Banach space X of dimension > 1 have a closed invariant subspace? The answer is false due to Enflo, Read. The answer is unknown for Hilbert spaces.

Proposition 5.27. *H Hilbert; $T \in L(H)$; $Y \leq H$. Then T acts on $Y \implies T^*$ acts on $Y^\perp$. In particular, for T Hermitian, T acts on $Y \implies T$ acts on $Y^\perp$*

Proof. Compute. □

Corollary 5.28. *Let T be a Hermitian operator on an n-dimensional Hilbert space, then H has an orthonormal basis of eigenvectors.*

Proof. Inductive proof, c.f. Linear algebra. □

We aim to generalize this to infinite dimensions.

5.2 Spectral Theory of Compact Operators

X Banach, $T \in L(X)$. Recall that any limit of finite rank operators is compact.

Proposition 5.29. *X an infinite dimensional Banach space; $T \in L(X)$ compact. Then 0 is an approximate eigenvalue of T.*

Proof. Since $\dim X = \infty$, there exists (x_n) s.t. $\|x_n\| = 1$ with $\|x_n - x_m\| \geq 1$ for all $n \neq m$. Then there exists a subsequence (x_{n_i}) with (Tx_{n_i}) convergent. Then consider the sequence $x_{n_i} - x_{n_{i+1}}$. This shows that $0 \in \sigma_{ap}(T)$. □

Note that 0 needs not be an eigenvalue, e.g., $T(\sum_n x_n e_n) = \sum_n \frac{1}{2^n} x_n e_n$ on ℓ_2.

Proposition 5.30. *X Banach; $T \in L(X)$ compact; $\lambda \in \mathbb{C}$ with $\lambda \neq 0$. Then λ is an approximate eigenvalue $\implies \lambda$ is an eigenvalue.*

Proof. λ is an approximate eigenvalue of T, so there exists (x_n) with $\|x_n\| = 1$ with $Tx_n - \lambda x_n \to 0$ as $n \to \infty$, so there exists a subsequence x_{n_i} with $Tx_{n_i} \to y$ for some $y \in X$, so $\lambda x_{n_i} \to y$. In particular, $y \neq 0$. Then $x_{n_i} \to y/\lambda$ and $T(y/\lambda) - \lambda(y/\lambda) = 0$. □

Proposition 5.31. *X a Banach space; $T \in L(X)$ compact. Then every eigen space $E(\lambda)$ for $\lambda \neq 0$ is finite dimensional.*

Proof. If not, then $T(B_X) \supseteq T(B_{E(\lambda)}) = \lambda B_{E(\lambda)}$ which would contradicts compactness assumption. □

A more elaborate version is the following.

Lemma 5.32. X Banach space; $T \in L(X)$ compact. Then for all $\delta > 0$, T has only finitely many eigenvalues with $|\lambda| > \delta$.

Proof. Suppose $\lambda_1, \lambda_2, ...,$ are distinct eigenvalues of T with $|\lambda_n| > \delta$ for all n. For each n, there exists $x_n \in X$ s.t. $Tx_n = \lambda x_n$. Then x_n are linearly independent. Define $X_n = \langle x_1, ..., x_n \rangle$. There exists $y_n \in X_n$, $\|y_n\| = 1$ s.t. $d(y_n, X_{n-1}) = 1$ by Riesz's lemma.

We claim that Ty_n has no convergent subsequence.

Proof of claim. Suppose $y_n = c_1x_1 + \cdots + c_nx_n$ for some $c_i \in \mathbb{C}$. Then $Ty_n = \lambda_1 c_1 x_1 + \cdots + \lambda_n c_n x_n$, so $Ty_n \in X_n$, so $d(Ty_n, X_{n-1}) = d(\lambda_n y_n, X_{n-1}) = |\lambda_n|$. For any $m < n$, we now have $Ty_m \in X_m$ and $d(Ty_n, X_{n-1}) = |\lambda_n| > \delta$, so $\|Ty_n - Ty_m\| > \delta$. □

This contradicts the assumption that T is compact. □

Theorem 5.33 (Spectral Theorem for Compact Operators)**.** *X Banach; $T \in L(X)$ compact. Then*

(i) Either $\sigma(T)$ is finite or $\sigma(T) = \{0, \lambda_1, \lambda_2, ...\}$ where $\lambda_n \to 0$

(ii) $\lambda \in \sigma(T)$; $\lambda \neq 0 \implies \lambda$ is an eigenvalue of T.

(iii) $\lambda \in \sigma(T), \lambda \neq 0 \implies \dim E(\lambda) < \infty$

Proof. We have already proved (iii). Note that (ii) implies (i). If $\sigma(T)$ is infinite, then any non-zero value in $\sigma(T)$ is an eigenvalue. For each $\delta > 0$, only finitely many e-values can have modulus larger than δ. So the eigenvalues form a sequence $\to 0$. $0 \in \sigma(T)$ since $\sigma(T)$ is closed.

To prove (ii), we show that $\forall \delta > 0$, $\lambda \in \sigma(T); |\lambda| > \delta \implies \lambda$ is an eigenvalue of T. Have $\partial\sigma(T) \cap \{z \in \mathbb{C} : |z| > \delta\} \subseteq \sigma_{ap}(T) \cap \{z \in \mathbb{C} : |z| > \delta\} \subseteq \{\lambda_1, ..., \lambda_k\}$ So $\sigma(T) \cap \{z \in \mathbb{C} : |z| > \delta\} = \{\lambda_1, ..., \lambda_k\}$. □

We now consider compact Hermitian operators.

Example 5.34. Suppose $T : \ell_2 \to \ell_2$ is defined by $T(\sum_n x_n e_n) = \sum_n \lambda_n x_n e_n$ where λ_n real and $\lambda_n \to 0$. This is clearly Hermitian and compact (being the limit of finite rank operators). We will show that this is the only example.

Theorem 5.35 (Spectral Theorem for Compact Hermitian Operators)**.** *H separable Hilbert space; $T \in L(H)$ compact Hermitian. Then*

(i) H has an orthonormal basis (e_n) consisting of eigenvectors of T.

(ii) The corresponding eigenvalues $\lambda_n \to 0$ if $\dim H = \infty$.

Remark 30. Note that (i) $\implies$ (ii). Using a proposition we proved earlier.

Proof. Let the eigenvalues of T be $\lambda_1, \lambda_2, ...$ (could be finite or even empty). Then the eigenspaces $E(\lambda_i)$ are pairwise orthogonal. Pick an orthonormal basis for each $E(\lambda_i)$, then their union is an orthonormal sequence which has closed linear span Y. We claim that $Y = H$.

Proof of claim. T acts on $E(\lambda_n)$ for each n, so T acts on $E(\lambda_n)^\perp$, so T acts on $\bigcap_n E(\lambda_n)^\perp = Y^\perp$. Now $T|_{Y^\perp}$ is compact since $Y^\perp$ is closed and T is compact, and $T|_{Y^\perp}$ has no eigenvalues. Therefore, $r(T|_{Y^\perp}) = 0$, i.e., $T|_{Y^\perp} = 0$, so $Y^\perp = \{0\}$ because 0 is not an eigenvalue of $T|_{Y^\perp}$. □

□

Theorem 5.36. *H Hilbert; $T \in L(H)$ compact Hermitian. Then there exists a closed subspace Y of H and an orthonormal basis (e_n) of Y and (λ_n) in $\mathbb{R}$ s.t. $\forall x \in H$, $x = \sum_n x_n e_n + z$; $z \in Y^\perp \implies Tx = \sum_n \lambda_n x_n e_n$.*

Proof. Let the non-zero eigenvalues of T be $\lambda_1, \lambda_2, ...$ (could be finite or even empty). Pick an orthonormal basis of each $E(\lambda_i)$ ($\dim E(\lambda_i) < \infty$). Their union (e_n) is an orthonormal basis for $Y = \overline{\langle E(\lambda_i) : \lambda_i \in \sigma(T),\ \lambda \neq 0 \rangle}$. Note that T acts on $Y^\perp$ with $T|_{Y^\perp}$ having no non-zero eigenvalues, so $r(T|_{Y^\perp}) = 0$, so $\|T|_{Y^\perp}\| = 0$ (since $T|_{Y^\perp}$ is Hermitian), so $T|_{Y^\perp} = 0$. □