Riemann Surfaces

Kevin

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1 The Complex Plane

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1.1 Review of Complex Analysis

- A domain $U \subseteq \mathbb{C}$ is an open (non-empty) and connected (hence path-connected) subset.
- Disk centered at z_0 with radius r: $D(z_0, r)$.
- Punctured disk $D'(z_0, r)$.

If $f : U \to \mathbb{C}$ is complex-diff, then $\det J = |f'|^2$, where J is the Jacobian of f regarded as a function on $\mathbb{R}^2$.

Definition 1.1. $f : U \to \mathbb{C}$, U domain, is holo'c on U if f is $\mathbb{C}$-diff for all $z_0 \in U$.

Theorem 1.2 (Taylor). *If f is holo'c on U and $D(z_0, r) \subseteq U$, then $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges on $D(z_0, r)$ with $a_n = f^{(n)}(z_0)/n!$. Then f holo'c on U implies that $f^{(n)}$ holo'c on U.*

Theorem 1.3 (Identity theorem). *f, g be holo'c on a domain U, then either $f \equiv g$ or for all $z_0 \in U$ there exists $r > 0$ s.t. $f(z) \neq g(z)$ for all $z \in D'(z_0, r) \subseteq U$.*

Recall that if $f : D'(z_0, r) \to \mathbb{C}$ is holo'c then z_0 is an isolated singularity. In this situation we have the (unique) Laurent expansion at z_0, $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ on $D'(z_0, r)$.

Theorem 1.4. *Exactly one occurs*

- *Removable*
- *Pole*
- *Isolated essential.*

Definition 1.5. $f : U \to \mathbb{C}$ (U domain) meromorphic if it's holo'c away from poles.

Theorem 1.6 (Open mapping theorem). *f holo'c on a domain U. Then f is either constant or an open map.*

Corollary 1.7 (Inverse function theorem). *If f is holo'c on a domain, then*

1. *f injective $\implies$ f^{-1} is holo'c with $(f^{-1})'(z) = 1/f'(f^{-1}(z))$.*
2. *if $f'(z_0) \neq 0$ for some $z_0 \in U$, then there exists an open nbd (nbd will always be open in this course) N of z_0 s.t. $f : N \to f(N)$ is biholomorphic (i.e., f and f^{-1} are both holo'c).*

1.2 Continuation of Power Series

Notation: $\Delta := D(0,1)$, *and* $T = \{z \mid |z| = 1\}$.

Suppose we have power series with radius of convergence R (not 0 or ∞) so $f : D(z_0, R) \to \mathbb{C}$ holo'c. WLOG, assume $R = 1$ and power series as domain Δ.

Definition 1.8. A point $z \in T$ is regular if $\exists D(z,r)$ and $g : D(z,r) \to \mathbb{C}$ holo'c s.t. $f \equiv g$ on $\Delta \cap D(z,r)$. Otherwise, we say that z is singular.

So z being regular means that f can be extended holomorphically to $\bar{f}$ on $\Delta \cup D(z,r)$.

Example 1.9. (i) $f(z) = \sum_n z^n$ diverges on T but the $f(z) = 1/(1-z)$ means that $T \setminus \{1\}$ are regular points

(ii) Take $f(z) = \sum_{n\geq 2} \frac{z^n}{n(n-1)}$ is absolutely conv. on T. Note that 1 is singular. If it's regular, then f'' extends but f'' is the geometric series.

Remark 1. (i) The set of regular points is open in T.

(ii) For $w \in \Delta$, let $\rho(w)$ be radius of convergence of the Taylor series at w. For $z \in T$, have z regular if and only if $\rho(z/2) > 1/2$.

Theorem 1.10. *If* $f(z) = \sum_{n\geq 0} a_n z^n$ *has r.o.c.* 1, *then there is a singular point on* T.

Proof. If not, then for all $w \in T$, $\exists r_w > 0$ s.t. f can be extended holomorphically to f_w on $\Delta \cup D(w, r_w)$. Note that $D(w, r_w) \cap D(w', r_{w'})$ is convex (if non-empty). Let $F : \Delta \cup \bigcup_{w\in T} D(w, r_w) \to \mathbb{C}$ be the extension of f. F is well-defined. If $z \in D(w, r_w) \cap D(w', r_{w'})$ then $f_w, f_{w'}$ both holo'c on $D(w, r_w) \cap D(w', r_{w'}) \cup \Delta$. By identity theorem, they agree. Now, we note that the domain of F includes $D(o, R)$ for some $R > 1$. (If not, take z_n with $1|z_n| < 1 + 1/n$) and $z_n \notin \text{dom}(F)$, then by sequential compactness, get a contradiction.) Contradiction with r.o.c being 1. ☐

Corollary 1.11. *If* $a_n \geq 0$ *for all* n, *then* 1 *is singular.*

Proof. For $z \in T$ and each k, we have $f^{(k)}(z/2) = \sum_{n=k}^{\infty} n(n-1)...(n-k+1)a_n(z/2)^{n-k}$. So $|f^{(k)}(z/2)| \leq |f^{(k)}(1/2)|$, so $\rho(z/2) \geq \rho(1/2)$. If $\rho(1/2) > 1/2$, then all $z \in T$ is regular. Contradiction. ☐

1.3 Complex Logarithm

Notation: $L_\theta = \{re^{i\theta} : r \geq 0\}$.

Definition 1.12. For any $\alpha \in \mathbb{R}$, define $\log_\alpha : \mathbb{C} \setminus L_{\alpha+\pi} \to \mathbb{C}$ as $\log_\alpha(re^{i\theta}) = \ln r + i\theta$, where $\theta \in (\alpha - \pi, \alpha + \pi)$.

1.4 Analytic Continuation (Plane)

Let D be a fixed domain of $\mathbb{C}$.

Definition 1.13. A function element of D is a pair (f, U), where U is a subdomain ($\subseteq D$) and f is a holo'c function defined on U.

Definition 1.14. Let (f, U) and (g, V) be function elements of D. Say (g, V) is a direct analytic continuation of (f, U) if $U \cap V \neq \varnothing$ and $f \equiv g$ on $U \cap V$.

Remark 2. The relation of being direct analytic continuation is relfexive and symmetric but not transitive. For instance, $f = \log_{\pi/2}$ on its domain, $g = \log_0$ on the right half plane, $h = \log_{-\pi/2}$ on its domain.

Definition 1.15. (g, V) is an analytic continuation of (f, U) along some path γ if there exists function elements $(f, U) = (f_1, U_1), (f_2, U_2), ..., (f_n, U_n) = (g, V)$ and a dissection $\{t_0 = 0 \leq t_1 \leq ... \leq t_n = 1\}$ s.t. (f_{i+1}, U_{i+1}) is a direct analytic continuation of (f_i, U_i) and $\gamma([t_{i-1}, t_i]) \subseteq U_i$. Write $(f, U) \approx (g, V)$.

An equivalence class $\mathfrak{F}$ is called a complete analytic function of D.

Note that if (g, V) and (h, V) are both analytic continuations of (f, U) along the **same** path γ, then $g \equiv h$ on V. (See ES1 or later).

Remark 3. Can do the same for meromorphic continuation.

1.5 INFORMAL Examples of Riemann Surfaces

1) Take disjoint copies $\mathbb{C}_k^*$ of $\mathbb{C}^*$ indexed by $k \in \mathbb{Z}$,so $(z,k) \in \mathbb{C}_k^*$. On $\mathbb{C}_k^*$ define $f_k(z) = \ln r + i\theta$ for $z = re^{i\theta}$ and $(2k-1)\pi < \theta \leq (2k+1)\pi$. Glue different copies along the cut. Get a space S. Then can define $F : S \to \mathbb{C}$ given by $(z,k) \mapsto f_k(z)$. F is cts, bijective and left invertible. Can do the same for $z^{1/n}$ except get n copies of $\mathbb{C}^*$.

3) $\sqrt{z^2-1}$. Cut $[-1,1]$. Take two copies of "cut complex plane", P_0, P_1. Glue.

2) What's the difference between a piano, a fish and a pot of glue? (?????????????????????????)

Ans You can tune a piano but you can't tuna fish!!!

1.6 Abstract Riemann Surfaces

Definition 1.16. A Riemann Surface R is a (non-empty) connected Hausdorff top. space with a collection of charts (homeo to open subsets of $\mathbb{C}$) such that the transition functions are holo'c.

Note the transition function $\tau_{ij} = \varphi_i \varphi_j^{-1}$ is biholo'c. In particular, any Riemann surface is orientable. Second countability is automatic from the definition. (Rado 1925)

Two atlases are said to be compatible (Same Riemann surface) if their union is an atlas.

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Proposition 1.17. *The Riemann sphere $\mathbb{C}_\infty$ is the top. space $\mathbb{C} \cup \{\infty\}$ (open sets are open subsets of $\mathbb{C}$ if $\infty \notin U$ or U^c is compact in $\mathbb{C}$ if $\infty \in U$.) Charts $\varphi_0 : \mathbb{C} \to \mathbb{C}$ and $\varphi_1(\mathbb{C}_\infty \setminus \{0\})$*

Given $P(z,w) \in \mathbb{C}[z,w]$ Consider $S = \{(z,w) \in \mathbb{C}^2 : P(z,w) = 0, (\partial P/\partial z, \partial P/\partial w) \neq (0,0)\}$. Then each conn. component is a Riemann surface. Suppose $(z_0, w_0) \in S$ and $\partial P/\partial z \neq 0$ at z_0. Get $f_{w_0} = P(-, w_0)$. By shrinking domain, assume $f_{w_0} D(z_0, r) \to \operatorname{im}(f_{w_0})$ Let $\gamma = \partial D(z_0, r)$ oriented counterclockwise.

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2 Maps on Riemann Surfaces

2.1 Analytic Maps

Definition 2.1. For Riemann Surfaces R, S, a cts function $f : R \to S$ is analytic if for all charts (φ, U) on R and (ψ, W) on S, have (if $U \cap f^{-1}(W) \neq \varnothing$) $\psi f \varphi^{-1} : \varphi(U \cap f^{-1}(W)) \to \mathbb{C}$ is holomorphic.

Note that if $R = S$ as top. spaces and $f = \mathrm{id}$, then two atlases are compatible iff id is analytic.

Consider $\bar{\mathbb{C}}$ ($\mathbb{C}$ with chart given by conjugation), then any biholomorphic map $f : \mathbb{C} \to \mathbb{C}$ is not analytic as a map $f : \bar{\mathbb{C}} \to \mathbb{C}$, but the map $f(z) = \bar{z}$ is analytic.

Let U be a domain in $\mathbb{C}$. A cts function $f : U \to \mathbb{C}_\infty$ is analytic iff f is holo'c on $U \setminus f^{-1}(\{\infty\})$ and $g = 1/f(z)$ is holo'c on $U \setminus f^{-1}(\{0\})$. If g is never 0, then f is holo'c. Otherwise $|f(z)| \to \infty$ near the zeros of g, i.e., f has poles, i.e., f is meromophic on U.

A map $f : \mathbb{C}_\infty \to \mathbb{C}$ is analytic iff $f : \mathbb{C} \to \mathbb{C}$ and $f(1/z)$ are both holomorphic. By compactness, f is bounded, so f is constant by Liouville.

Definition 2.2. Let R be a Riemann surface. An analytic function f on R is $f : R \to \mathbb{C}$ (analytic), and a meromorphic function g on R is $g : R \to \mathbb{C}_\infty$ (analytic)

Proposition 2.3. *$f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is analytic and nonconstant iff*

$$f(z) = \frac{c(z-\alpha_1)\cdots(z-\alpha_m)}{(z-\beta_1)\cdots(z-\beta_n)}$$

where $c \neq 0$ and $\alpha_i \neq \beta_j$ with $\infty \mapsto \begin{cases} \infty & m > n \\ c & m = n \\ 0 & m < n \end{cases}$

Proof. Analytic iff $f(z)$ and $f(1/z)$ are both meromorphic on $\mathbb{C}$.

($\Leftarrow$): Clear ($\Rightarrow$): If $f(\infty) \neq \infty$. Replace by $1/f$. By continuity, there exists $K > 0$ s.t. no poles in $|z| > K$, so we have a finite number of poles $\{\zeta_1, ..., \zeta_i\}$. Have Laurent series around each pole. Let $Q_j(z)$ be the principal part of the Laurent series around ζ_j. Let $\bar{f} = f - Q_1 - \cdots - Q_i$, then f has removable singularities at each ζ_j and holo'c elsewhere (including ∞). So $\bar{f} : \mathbb{C}_\infty \to \mathbb{C}$ is analytic, so constant. □

Theorem 2.4 (Identity theorem for Riemann surfaces). *If $f, g : R \to S$ are analytic maps between Riemann surfaces, then $f \equiv g$ on R or $\forall w \in R$, $\exists \mathcal{N}$ open nbd of w s.t. $f \neq g$ on $\mathcal{N} \setminus \{w\}$.*

Proof. Let $E = \{w \in R : \exists \mathcal{N} \ni w,\ f \equiv g \text{ on } \mathcal{N}\}$, $F = \{w \in R : \exists \mathcal{N} \ni w,\ f \neq g \text{ on } \mathcal{N} \setminus \{w\}\}$. Need to show $E \cup F = R$.

If $f(w) \neq g(w)$. Find U, V open and separating $f(w), g(w)$, then $\mathcal{N} = f^{-1}(U) \cap f^{-1}(V)$ shows $w \in F$.

If $f(w) = g(w)$. Take charts (φ, U) in R around w and (ψ, W) in S around $f(w)$. $\psi f \varphi^{-1}$ and $\psi g \varphi^{-1}$ are holo'c on $\varphi(U \cap f^{-1}(U) \cap g^{-1}(V))$ and agree at $\varphi(w)$. By identity theorem in CA, there is a disk $D(\varphi(w), r)$ s.t. $\psi f \varphi^{-1}, \psi g \varphi^{-1} \varphi^{-1}$ either agree or disagree except at $\varphi(w)$. Let $\mathcal{N} = \varphi^{-1}(D(\varphi(w), r))$, so $w \in E \cup F$.

We are done by connectedness. □

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Theorem 2.5 (Open Mapping Thm for Riemann surfaces). *If $f : R \to S$ is a non-constant analytic map, then W open in $R \implies f(W)$ is open in S.*

Proof. Take $w \in W$ and charts (φ, U), (ψ, V) around $w, f(w)$ resp. Let $N = \varphi(U \cap f^{-1} \cap W)$, which is open in $\mathbb{C}$, and consider $\psi f \varphi^{-1}$. WLOG, assume N is connected, so by identity thm on $\mathbb{C}$, either const (contradiction) or $\psi f \varphi^{-1}(N)$ is open in $\psi(V)$, so $f\varphi^{-1}(N)$ is open in V. Let $M_w = \varphi^{-1}(N)$ which is a nbd of w in R with $M_w \subseteq W$ and $f(w) \in f(M_w)$ is open in S, so $f(w)$ is an interior point of $f(W)$. □

Corollary 2.6. *Let $f : R \to S$ be analytic with R compact. Then either f is surjective (so S is compact) or f is constant.*

2.2 Local Representation of Analytic Maps

Let $U \subseteq \mathbb{C}$ be a domain and let $f : U \to \mathbb{C}$ be a non-const holo'c function with $z_0 \in U$. Locally, $f(z) = f(z_0) + \sum_{k \geq 1} a_k (z - z_0)^k$. Let $m \geq 1$ be the smallest s.t. $a_m \neq 0$. Then $f(z) = f(z_0) + (z - z_0)^m g(z)$ where $g(z_0) \neq 0$.

Definition 2.7. The multiplicity of f at z_0 is $m_f(z_0) = m$ as above.

Note that the set of points with multiplicity > 1 form a discrete set. Also, the multiplicity is multiplicative (w.r.t. composition).

Theorem 2.8 (Local mapping thm for domains). *For non-const $f : U \to \mathbb{C}$ holo'c, $z_0 \in U$, $m = m_f(z_0)$. Then $\exists$ nbd N of $z_0 \in U$ and a biholomorphic $\beta : N \to D(0, \delta)$ (for arbitrarily small $\delta > 0$) s.t. $f(z) = f(z_0) + \beta(z)^m$ on N.*

Proof. Write $f(z) = f(z_0) + (z - z_0)^m g(z)$ on some $D(z_0, r) \subseteq U$ w/ g holo'c and non-zero at z_0. Write $g(z_0) = re^{i\alpha} \neq 0$. Have a holo'c branch $\log_\alpha$. Define $g^{-1}(\mathbb{C} \setminus L_{\pi+\alpha} \cap D(z_0, r))$ is a nbd of z_0 on which g is holo'c. Define $\beta(z) = (z - z_0)e^{1/m} \log_\alpha g(z)$. Can see that $\beta'(z_0) \neq 0$. By IFT, have nbd $N'' \subseteq N'$ w/ $\beta : N'' \to \beta(N'')$ biholo'c. Take $\beta^{-1}(D(0, \delta))$ and restrict. □

Let $f : R \to S$ be analytic and $z_0 \in R$ with charts (φ_0, U_0), (ψ_0, V_0) around z_0 and $f(z_0)$ resp.

Definition 2.9. The multiplicity $m_f(z_0)$ is $m_{\psi_0 f \varphi^{-1}}(\varphi_0(z_0))$.

Lemma 2.10. *This is independent of charts.*

Proof. For another pair of charts (φ_1, U_1) and (ψ_1, V_1). Write $\psi_1 f \varphi_1^{-1} = (\psi_1 \psi_0^{-1})(\psi_0 f \varphi_0^{-1})(\varphi_0 \varphi_1^{-1})$. Done (multiplicative). □

2.3 Degree

Theorem 2.11 (Valency Thm)**.** *Let $f : R \to S$ be a non-const and analytic. Suppose R is compact. Then $\exists n \geq 1$ (degree/valency) s.t. $\forall w \in S$, $\# f^{-1}(w) = n$ (with multiplicity).*

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Proof. For all $w_0 \in S$, $f^{-1}(w_0)$ is compact and discrete so finite, say $\{z_1, ..., z_q\} \neq \varnothing$. For $A \subseteq R$, and $w \in S$, let $n_A(w) = \sum_{z \in f^{-1}(w) \cap A} m_f(z)$.

R Hausdorff, so have q disjoint nbds M_i of z_i. Choose charts (φ_i, U_i) around z_i and (ψ, V) around w_0. Wlog, $\psi(w_0) = 0$. Then $\psi f \varphi_i^{-1}$ is holo'c around $\varphi_i(z_i)$, so there exists a nbd H_i in $\mathbb{C}$ around $\varphi_i(z_i)$ s.t. $\psi f \varphi_i^{-1}(\zeta) = \psi f(z_i) + \beta_i(\zeta)^{m_f(z_i)}$ for some $\beta_i : H_i \to D(0, r_i)$ is biholo'c. Take $s = \min r_i$ and define $N_i = \varphi_i^{-1}(H_i)$. f is m_i-to-1 $N_i \to \psi^{-1}(D(0, s))$ and N_i are disjoint nbd of z_i in R.

Consider $R \setminus \bigcup_{i=1}^{q} N_i$. It's compact, so $M = S \setminus f(R \setminus \bigcup N_i)$ is open, so a nbd of w_0. Take any $w \in \bigcap f(N_i) \cap M$ (nbd of w_0) with $R = \bigsqcup N_i \bigsqcup (R \setminus \bigcup N_i)$. Have $n_R(w) = n_{N_1}(w) + \cdots + n_{N_q}(w) + n_{R \setminus \bigcup U_i}(w) = m_f(z_1) + \cdots + m_f(z_q) + 0$, so n_R is loc.const. □

Corollary 2.12. *$f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is biholo'c (bianalytic) iff f is a Möbius transformation.*

2.4 Harmonic Functions

Definition 2.13. A func $u : U \to \mathbb{R}$, U domain, is harmonic if $u \in C^2(U)$ and $\nabla^2 u = 0$.

Theorem 2.14. *u is harmonic on disk $D \implies \exists f : D \to \mathbb{C}$ holo'c with $u = \Re f$.*

Corollary 2.15. *If u is harmonic on a domain U, then u is smooth.*

Corollary 2.16. *If $U \to V$ holo'c on a domain and $u : V \to \mathbb{R}$ harmonic, then $u \circ g$ harmonic.*

Definition 2.17. R is a Riemann surface. $u : R \to \mathbb{R}$ is harmonic if $u\varphi_\alpha^{-1}$ is harmonic for all charts φ_α.

Theorem 2.18 (Identity Thm for harmonic functions)**.** *Let R be a Riemann surface with $u, v : R \to \mathbb{R}$ both harmonic, then either $u \equiv v$ or $\{z \in R : u(z) = v(z)\}$ has empty interior.*

Proof. ES1 □

Corollary 2.19 (Open Mapping for harmonic functions)**.** *Let R be a Riemann surface with a non-const harmonic function $R \to \mathbb{R}$, then u is an open map.*

Proof. Pick $w \in W$ open and chart (φ, U) at w. Wlog, assume $U \subseteq W$ and $\varphi(U)$ is a disk, so there exists a holo'c function $f : \varphi(U) \to \mathbb{C}$ s.t. $\Re f = u\varphi^{-1}$. Note that f is non-constant, so open mapping theorem for holo'c functions, $f(\varphi(U))$ is open. Project to the real part, deduce that $u(U)$ is open in $\mathbb{R}$, so $u(w)$ is an interior point of W for all $w \in W$. □

Corollary 2.20. *If $u : R \to \mathbb{R}$ is a harmonic function and R is compact, then u is const.*

3 Covering Maps

In this section, assume all topological spaces are Hausdorff, connected, and locally path-connected, unless stated otherwise.

3.1 Local Homeomorphisms

Definition 3.1. A map $f : X \to Y$ is a local-homeo if $\forall x \in X$, $\exists N_x$ open nbd of x s.t. $f(N_x)$ is open in Y and $f|_{N_x} : N_x \to f(N_x)$ is a homeo.

Lemma 3.2. *Let $f : R \to S$ be non-const and analytic.*

(i) If $m_f(z) \equiv 1$ then f is a local homeo.

(ii) If $Z = \{z \in R : m_f(z) > 1\}$, have $R \setminus Z$ RS so $f|_{R \setminus Z}$ is a local homeo

Proof. (i): For $z \in R$, pick charts $(\varphi, U), (\psi, W)$. Do stuff over $\mathbb{C}$ and shrink the domain.

(ii): Z is discrete. $\phi(z)$ has an accumulation point, then identity theorem on $\mathbb{C}$ and on riemann surfaces imply that f is const. □

3.2 Paths and Lifts

Definition 3.3. Lift of paths

Theorem 3.4. *For $f : \tilde{X} \to X$ local homeo and $\tilde{\gamma}, \tilde{\tilde{\gamma}}$ two lifts of γ s.t. $\tilde{\gamma}(0) = \tilde{\tilde{\gamma}}(0)$, then $\tilde{\gamma} \equiv \tilde{\tilde{\gamma}}$.*

Proof. Define $E = \{t :\in [0,1] : \tilde{\gamma}(t) = \tilde{\tilde{\gamma}}(t)\}$. E is closed since $\tilde{X}$ is Hausdorff and paths are cts. E is also open. Let $\tau \in E$. Pick nbd $\tilde{N}$ around $\tilde{\gamma}(\tau)$ s.t. $f|_{\hat{N}}$ is inj. For $t \in (\tau - \delta, \tau + \delta) \cap I$, $\tilde{\gamma}(t), \tilde{\tilde{\gamma}}(t) \in \hat{N}$. f is inj on $\hat{N}$, so $\tilde{\gamma}(t) = \tilde{\tilde{\gamma}}(t)$ □

Definition 3.5. Covering maps

Lemma 3.6. *The cardinality of the fiber is constant.*

Proof. Equiv rel: $x \sim x'$ iff $|\pi^{-1}(x)| = |\pi^{-1}(x')|$. $|\pi^{-1}(x)|$ is locally constant. Done by connectedness. □

Theorem 3.7. *For non-const $f : R \to S$ analytic and R cpt, the map $f : R \setminus f^{-1}f(Z) \to S \setminus f(Z)$ is a covering map of Riemann surfaces.*

Proof. Z is discrete and closed in a cpt set, so Z is finite, so $f(Z)$ is finite. For all $w \in S$, $f^{-1}(w)$ is finite (cpt discrete), so $f^{-1}f(Z)$ is finite. □

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3.3 Branched Covering Maps

Definition 3.8. If $\tilde{S}, S$ are Riemann surfaces, a branched covering map $p : \tilde{S} \to S$ is where $\forall s \in S$ there exists a nbd N of s in S and a homeo $\Phi : N \to \Delta$ (unit disk) with $\Phi(s) = 0$ s.t. $p^{-1}(N) = \bigsqcup_{i \in I} U_i$ a disjoint union of (connected) open subsets of $\tilde{S}$, each with a homeo $\Psi_i : U_i \to \Delta$ s.t. $\forall z \in \Delta$, $\Phi p \Psi_i^{-1} = z^{m_i}$ for some $m_i \in \mathbb{N}$.

A branch point s of p is a point $s = \Psi_i^{-1}(0) \in \tilde{S}$ where $m_i > 1$. (The unique point $x \in U_i$ with $p(x) = s$) A critical value of p is $p(x) \in S$ for any branch point x.

e.g., any non-const analytic map $f : R \to S$ for R compact Riemann surface. (follows from thm 3.7 and valency theorem thm 2.11)

Theorem 3.9. *Any compact orientable connected topological surface (without boundary) S is homeo to a genus g surface.*

Theorem 3.10 (Riemann-Hurwitz formula)**.** *Let $f : R \to S$ non-const analytic map between compact Riemann surfaces. Suppose f has deg n. Then*

$$\sum_{p \in R} (m_f(p) - 1) = 2(g_R - 1) - 2n(g_S - 1)$$

where g_R, g_S are genera of R, S resp.

Proof Sketch [Non-examinable.] Finite number of branch points $\{r_1, ..., r_k\}$ on R (where $m_f(r_i) > 1$). For any $w \in S$ not critical value, have $|f^{-1}(w)| = n$ by 2.11. Take a polygonal decomposition $\mathcal{D}$ of S (existence by Rado) including all critical values in the vertices. Then the preimage $f^{-1}(D)$ is a polygonal decomposition of R with nF faces, nE edges and $nV - \sum_{j=1}^{k}(m_f(r_j) - 1)$. Compute. □

In particular, $g_R \geq g_S$ (cf. ES2).

3.4 The (Topological) Monodromy Theorem

Theorem 3.11. *Let $\pi : \tilde{X} \to X$ be a covering map, γ path in X and $p \in \tilde{X}$ any pt with $\pi(p) = \gamma(0)$. Then there exists a lift $\tilde{\gamma}$ of γ with $\tilde{\gamma}(0) = p$ (unique by thm 3.4)*

Proof. (cf. Htpy lifting) □

Definition 3.12. Homotopy rel $\partial[0,1]$. (cf. algtop)

Definition 3.13. Simply connected (cf. algtop)

Theorem 3.14 (Topological Monodromy thm)**.** *Let $f : Y \to X$ be a local homeo. Suppose α, β are paths in X which are homotopic rel $\{0,1\}$. Take $y_0 \in Y$ with $f(y) = \alpha(0) = \beta(0) = x_0$. Suppose that any path γ in X with $\gamma(0) = x_0$ has a lift $\tilde{\gamma}(0) = y_0$. Then $\tilde{\alpha}, \tilde{\beta}$ are homotopic in Y rel $\{0,1\}$. In particular, $\tilde{\alpha}(1) = \tilde{\beta}(1)$.*

Proof omitted.

Corollary 3.15. *If $\pi : \tilde{X} \to X$ is a covering map and if X is simply connected, then π is a homeo.*

Proof. ES2. □

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4 Space of Germs

4.1 Abstract Analytic Continuation

In this section assume $D \subseteq \mathbb{C}$ is a fixed domain. Fix $w \in D$ and let $\mathfrak{F}_w = \{f_n$ elements (f, U) on D : $w \in U\}$.

Definition 4.1. For (f, U), and (g, V) in $\mathfrak{F}_w$ (so $w \in U \cap V$), say $(f, U) \sim_w (g, V)$ if $f = g$ on some nbd of w (but not necessarily on all of $U \cap V$). This is an equiv rel for each $w \in D$. The equiv class in $\sim_w$ containing (f, U) is the germ of f at w, denoted $[f]_w$.

Thus $[f_1]_{w_1} = [f_2]_{w_2}$ iff $w_1 = w_2$ and $f_1 = f_2$ on some nbd of w_1.

Definition 4.2. The space of germs $\mathfrak{G}(D)$ is the set of all germs $[f]_w$ over all (f, U) on D and all elements $w \in D$.

Given a function element (f, U) on D, we write $[f]_U = \{[f]_z : z \in U\} \subseteq \mathfrak{G}(D)$. (so it comprises one $\sim_z$ equiv class for each point $z \in U$)

Definition 4.3. The topology on $\mathfrak{G}(D)$ is generated by basis $[f]_U$ for (f, U) a function element on D, so $S \subseteq \mathfrak{G}(D)$ is open iff $S = \bigcup_{\alpha \in A} [f_\alpha]_{U_\alpha}$

This is indeed a basis. If V_1, V_2 open in $\mathfrak{G}(D)$ with $[f]_z \in V_1 \cap V_2$. Have U_i open in D and f_i holo'c on U_i s.t. $[f]_z$ is in $[f_i]_{U_i} \subseteq V_i$. So $z \in U_1 \cap U_2$ and $f = f_i$ on a nbd of N_i of z, so $[f]_{N_1 \cap N_2}$ is in $V_1 \cap V_2$

4.2 Topological Properties of $\mathfrak{G}$

Proposition 4.4. *$\mathfrak{G}(D)$ is Hausdorff.*

Proof. Suppose $[f]_z \neq [g]_w$.

If $z \neq w$ then take disjoint domains separating z, w, then $[f]_{U_1}$ and $[g]_{U_2}$ are disjoint open sets separating $[f]_z$, $[g]_w$.

If $z = w$ and $[h]_v \in [f]_W \cap [g]_W$, where $(f, W) \in [f]_w$ and $(g, V) \in [g]_w$ and $w \in W$ (wlog assume W is a domain by shrinking). Then $[h]_v = [f]_v = [g]_v$, so $f = g$ on a nbd of v and $v \in W$, so $f = g$ on W, and W is a nbd of w. Contradiction. □

Definition 4.5. The projection map $\pi : \mathfrak{G}(D) \to D$ is defined by $\pi([f]_z) = z$.

Proposition 4.6. *π is a local homeo.*

Proof. For (f,U) on D, consider the restriction $\pi : [f]_U \to U$. π is open [If V is open $[f]_U$, then $V = \bigcup_\alpha [f_\alpha]_{U_\alpha}$, $\pi(V) = \bigcup_\alpha U_\alpha$]. π is bijective. π is cts if U open in D then $\pi^{-1}(U) = \bigcup_\alpha [f_\alpha]_{U_\alpha}$, where f_α is any holo'c function on any subdomain of U. □

Can put a Riemann surface structure on (conn. components of) $\mathfrak{G}(D)$. Charts (φ, V) for $V = [f]_U$ basic open set and $\varphi = \pi|_V : V \to U$. Transition maps id, so π is analytic.

Definition 4.7. The evaluation map $\epsilon : \mathfrak{G}(D) \to \mathbb{C}$ is $\epsilon([f]_z) = f(z)$.

This is well-defined and analytic as it is $f \circ \pi$ on $[f]_U$.

Theorem 4.8. *Let (f,U) and (g,V) are function elements on D. Let γ be a path in D with $\gamma(0) \in U$, $\gamma(1) \in V$. Then (g,V) is an analytic continuation of (f,U) along γ iff there exists a path $\tilde{\gamma}$ in $\mathfrak{G}(D)$ with $\pi \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}(0) = [f]_{\gamma(0)}$, $\tilde{\gamma}(1) = [g]_{\gamma(1)}$.*

Proof. ($\Rightarrow$:)Have a chain of direct analytic continuation $(f_1,U_1), ..., (f_n,U_n)$ with $0 = t_0 \le \cdots \le t_n = 1$ with $\gamma([t_{j-1},t_j]) \subseteq U_j$. Define $\tilde{\gamma}$ on $[t_{j-1},t_j]$ by $\tilde{\gamma}(t) = [f_j]_{\gamma(t)}$. $f_{j-1} = f_j$ on a nbd of $\gamma(t_j)$. $\tilde{\gamma}$ is cts on $[t_{j-1},t_j]$. [If $[h]_W$ is an open set, then $t \in \tilde{\gamma}^{-1}[h]_W$ iff $\gamma(t) \in W$ and $f_j = h$ on some nbd of N of $\gamma(t)$. γ is cts, so there is an open interval I in $[t_{j-1},t_j]$ around t with $\gamma(I) \subseteq W \cap N$, so $I \subseteq \tilde{\gamma}^{-1}[h]_W$]. Can check directly that this is a lift.

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($\Leftarrow$:) $\tilde{\gamma}[0,1]$ is compact. Cover by finitely basic open sets $[f_1]_{U_1}, ..., [f_n]_{U_n}$ with U_i disks. Get $0 = t_0 \le \cdots \le t_n = 1$ with $\tilde{\gamma}[t_{j-1},t_j] \subseteq [f_j]_{U_j}$. Have $\tilde{\gamma}(0) = [f_1]_{\gamma(0)}$. $f = f_1$ on a nbd of $\gamma(0)$. Similarly, $g = f_n$ near $\gamma(1)$. $\tilde{\gamma}(t_j) \in [f_j]_{U_j} \cap [f_{j+1}]_{U_{j+1}}$ so same argument applies. So $(f_1,U_1), (f_2,U_2), ..., (f_n,U_n)$ direct analytic continuations. and $\gamma[t_{j-1},t_j] = \pi\tilde{\gamma}[t_{j-1},t_j] \subseteq \pi[f_j]_{U_j} = U_j$ all j. □

Corollary 4.9. *If $(g,V), (h,V)$ are analytic continuations of (f,U) along the same path γ, then $g \equiv h$ on V.*

Proof. Get two lifts with the same starting point. The projection $\mathfrak{G}(D) \to D$ is a local homeo, so must have $[g]_{\gamma(1)} = [h]_{\gamma(1)}$, i.e, $g = h$ on a nbd of $\gamma(1)$. □

Corollary 4.10 (Classical Monodromy thm)**.** *Suppose D is a simply connected domain and (f,U) a function element on D which can be analytically continued along every path in D starting in U. Take $(g,V), (h,V)$ analytic continuation along paths α, β in D with the same endpoints. Then $g \equiv h$ on V.*

Proof. Have lifts $\tilde{\alpha}, \tilde{\beta}$. Lift the path homotopy $\alpha \simeq \beta$ to a path homotopy of $\tilde{\alpha} \simeq \tilde{\beta}$. So $g = h$ on a nbd of $\alpha(1) = \beta(1)$. □

Corollary 4.11. *For any complete analytic function $\mathfrak{F}$, let $\mathfrak{G}_\mathfrak{F} = \bigcup_{(f,U)\in\mathfrak{F}} [f]_U \subseteq \mathfrak{G}(D)$. These are exactly the (path) components of $\mathfrak{G}(D)$.*

Proof. $\mathfrak{G}_\mathfrak{F}$ is path conn by thm. Each $\mathfrak{G}_\mathfrak{F}$ is open and they partition $\mathfrak{G}(D)$. □

So $\mathfrak{G}_\mathfrak{F}$ is a Riemann surface.

An analytic continuation along a path is equiv to a path in $\mathfrak{G}_\mathfrak{F}$ lifting the path. The eval map $\varepsilon : \mathfrak{G}_\mathfrak{F} \to \mathbb{C}$ is analytic. FOr fn elt (f,U). ε is a single-valued extension of $f \circ \pi$ to all of $\mathfrak{G}_\mathfrak{F}$.

5 Periodic Functions

5.1 Periods

Definition 5.1. Let $f : \mathbb{C} \to X$ be any fn to a set. A period $\omega \in \mathbb{C}$ of f s.t. $f(z+w) = f(z)$ for all $z \in \mathbb{C}$. Ω_f the set of all periods of f.

Ω_f is an additive subgroup of $\mathbb{C}$. If X is Hausdorff and f is cts, then Ω_f is closed in $\mathbb{C}$.

Lemma 5.2. *For non-const analytic map $f : \mathbb{C} \to S$ to a Riemann surface, the set Ω_f consists of isolated points.*

Proof. Let $g(z) = f(0)$. If $w = \alpha$ where $\alpha \in \Omega_f$ is an accumulation point, then have a sequence $w_i \to \alpha$, so $f(w_i) \to f(0)$, so f is const by identity theorem. □

Theorem 5.3. *Let $P \subseteq \mathbb{C}$ be an additive subgroup consisting of isolated pts. Then*

(0) $P = \{0\}$

(1) $P = \mathbb{Z}\langle\omega\rangle$ *for some* $\omega \in \mathbb{C}^\times$

(2) $P = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ *for* $\omega_i \in \mathbb{C}^\times$ *and* ω_i *are* $\mathbb{R}$*-linearly independent, i.e.,* $w_2/w_1 \notin \mathbb{R}$.

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Proof. Suppose $P \neq \{0\}$. P is closed in $\mathbb{C}$. Pick $\omega_1 \in P \setminus \{0\}$ with minimal modulus. If $P = \mathbb{Z}\omega_1$, then done. Otherwise, choose $\omega_2 \in P \setminus \mathbb{Z}\omega_1$ with minimal modulus. If $\omega_2 \in \mathbb{R}\omega_1$, then $\omega_2 = (k+\delta)\omega_1$ for some $k \in \mathbb{Z}$ and $\delta \in [-1/2, 1/2]$, then $\omega_2 - k\omega_1 = \delta\omega_1 \in P$ and $|\delta\omega_1| < |\omega_1|$, so $\delta = 0$ by minimality. So ω_1, ω_2 is $\mathbb{R}$-linearly indep.

Now for any $\omega \in P$, write $\omega = \lambda_1\omega_1 + \lambda_2\omega_2$. As before, can write the coeff as some integer plus something in $[-1/2, 1/2]$, so wlog assume $\lambda_1, \lambda_2 \in [-1/2, 1/2]$. Then $|\omega| \leq |\lambda_1||\omega_1| + |\lambda_2||\omega_2| \leq |\omega_2|$, but $\omega \in P$, so $\omega \in \mathbb{Z}\omega_1$. □

5.2 Simply Periodic Functions

Let $f : \mathbb{C} \to X$ analytic with $\Omega_f = \mathbb{Z}\omega$, $\omega \neq 0$. Then $g(z) = f(z\omega)$ has periods $\Omega_g = \mathbb{Z}$.

Theorem 5.4. *Let* $q : \mathbb{C} \to \mathbb{C}^\times$ *be* $q(z) = e^{i2\pi z}$. *Then given analytic function* $f : \mathbb{C} \to \mathbb{C}_\infty$ *with* $\Omega_f = \mathbb{Z}$, *there exists a unique analytic function* $\bar{f} : \mathbb{C}^\times \to \mathbb{C}_\infty$ *s.t.* $f = \bar{f} \circ q$

Proof. Set $\bar{f}(e^{i2\pi z}) = f(z)$. Well defined. For any $w \in \mathbb{C}^\times$, can define log in a nbd of w on which $\bar{f}(w) = f((\log w)/2\pi i)$, so $\bar{f}$ is analytic. □

Corollary 5.5. *With f as in thm 5.4. Suppose f has no poles on some horizontal strip* $S = \{z : \alpha < \Im z < \beta\}$. *Then* $f(z) = \sum_{k=-\infty}^{\infty} a_k e^{i2\pi kz}$, *which converges loc. uniformly on S.*

Proof. Find $\bar{f}$ as in thm 5.4. which has no poles on $q(S)$ which is an annulus. Get Laurent expansion for $\bar{f}$ which is loc. unif. conv. Substitute. □

5.3 Doubly Periodic Functions

Definition 5.6. An elliptic function $f : \mathbb{C} \to \mathbb{C}_\infty$ is an analytic function with set of periods Ω_f of type (2) thm 5.3 (or constants). Such a function is determined by its values on a fundamental parallelogram. $\rho = \{z + t_1\omega_1 + t_2\omega_2 : t_1, t_2 \in [0, 1)\}$ for $z \in \mathbb{C}$. ($P = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$)

Lemma 5.7. *Any nonconst elliptic function has at least one pole.*

Proof. If not, then $f : \mathbb{C} \to \mathbb{C}$ holo'c and bounded on a fundamental parallelogram, so constant by Liouville. □

Definition 5.8. A lattice $L \subseteq \mathbb{C}$ is a discrete additive subgroup of $\mathbb{R}^2$, i.e., P of type (2) in 5.3.

Definition 5.9. Given a lattice L, the $L-$torus is the Riemann surface $\mathbb{C}/L$.

Note that $q : \mathbb{C} \to \mathbb{C}/L$ is a covering map and is analytic.

Note that the map $f : S^1 \times S^1 \to \mathbb{C}/L$, $(e^{i2\pi\theta_1}, e^{i2\pi\theta_2}) \mapsto q(\theta_1\omega_1 + \theta_2\omega_2)$. So $\mathbb{C}/L$ nad $\mathbb{C}/L'$ are homeomorphic as top. spaces but need not be conformally equivalent.

Theorem 5.10. *Let* $q : \mathbb{C} \to \mathbb{C}/L$ *be as above. Given elliptic function* $f : \mathbb{C} \to \mathbb{C}_\infty$ *with* $\Omega_f \supseteq L$, *then there exists a unique analytic* $\bar{f} : \mathbb{C}/L \to \mathbb{C}_\infty$ *s.t.* $f = \bar{f} \circ q$.

(cf. thm 5.4)

Proof. Let $\bar{f}(q(z)) = f(z)$. This is well-defined by periodicity. $\bar{f}$ is cts since for $V \subseteq \mathbb{C}_\infty$ open, $q^{-1}\bar{f}^{-1}(V) = f^{-1}(V)$. $\bar{f}$ is analytic since on a chart $(q|_U)^{-1}$ for suff small U, have $\bar{f} \circ (q|_U)^{-1}$ analytic. Uniqueness: if $\bar{\bar{f}}$ is another choice, then $\bar{\bar{f}}[z] = \bar{f}[z]$ is forced. □

Corollary 5.11. *There exists $n \geq 2$ s.t. f maps each fundamental parallelogram ρ_L, n-to-1 to $\mathbb{C}_\infty$.*

Proof. Have nonconst $\bar{f}$ from $\mathbb{C}/L$ compact in □

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5.4 Weierstrass $\wp$-functions

Definition 5.12. For lattice Ω, the Weierstrass $\wp$-function is given by

$$\wp_\Omega(z) = \frac{1}{z^2} + \sum_{\omega \in \Omega \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

Lemma 5.13. *For $t > 0$, $\sum_{\omega \in \Omega \setminus \{0\}} \frac{1}{|\omega|} t$ converges iff $t > 2$.*

Proof. Let $\Omega = \langle \omega_1, \omega_2 \rangle$, ω_1, ω_2 indep. Let $S = \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 : |\lambda_1| + |\lambda_2| = 1\}$ (cpt) The map $(\lambda_1, \lambda_2) \mapsto (\lambda_1 \omega_1 + \lambda_2 \omega_2)$ is cts and non-zero on S, so there exists $0 < c \leq C$ s.t. $\forall (k, l) \in \mathbb{Z}^3 \setminus \{(0,0)\}$ $c(|k| + |l|) \leq |k\omega_1 + l\omega_2| \leq C(|k| + |l|)$. $[\lambda_1 = k/(|k| + |l|)]$. So $\sum_{(k,l) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{|k\omega_1 + l\omega_2|}$ conv iff $\sum_{(k,l) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{(|k|+|l|)^t}$ conv iff $\sum_{m=1}^\infty \sum_{(k,l) \in \mathbb{Z}^2 \setminus \{(0,0)\}, \|(k,l)\|_1 = m} \frac{1}{(|k|+|l|)^t}$ conv iff $\sum_{m=1}^\infty 4m \frac{1}{m^t}$ conv. iff $t > 2$. □

Theorem 5.14. *The function $\wp_\Omega$ is meromorphic, has period set Ω, egen, and 2-to-1 on $\mathbb{C}/\Omega$.*

Proof. Fix $r > 0$ and let $z \in D(0, r)$. Estimate the tail when $\omega \in \Omega$ with $|\omega| \geq 2r$:

$$\left| \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right| \leq \frac{\frac{r|\omega|}{2} + 2r|\omega|}{|\omega^4||z/\omega - 1|^2} \leq \frac{5r}{2|\omega|^3} \frac{1}{1/4} = \frac{10r}{\omega^3}$$

So $\wp_\Omega$ converges locally (abs) uniformly, so meromorphic (loc. unif. lim of meromorphic function).

Differentiate term by term. Get $\wp_\Omega(z) = -2 \sum_{\omega \in \Omega} \frac{1}{(z-\omega)^3}$ which is elliptic with period set including Ω. Consider $\wp_\Omega(z + \omega_1) = \wp_\Omega(z) + c$ Let $z = -\omega_1/2$, we see that $c = 0$ as the function is even. By considering poles, see that Ω contains the set of period.

Note that 0 is a double pole, so on $\mathbb{C}/\Omega$ has degree 2. □

Note that $\wp_\Omega$ has a branch pt at 0 and at the zeros of $\wp'_\Omega$. $\wp'_\Omega$ has a triple pole only at $\omega \in \Omega$, so degree 3 on $\mathbb{C}/\Omega$. Odd function, so have zeros at $\omega_1/2$, $\omega_2/2$ and $(\omega_1 + \omega_2)/2$. They are simple since $\wp'_\Omega$ has deg 3. Can check Riemann-Hurwitz formula.

6 Uniformization

(Owen's signature)

Let $\tilde{X}, X$ be topological spaces. Recall covering space (I have recalled pls hurrry up.)

Theorem 6.1. *For X conn and loc. simp. conn, there is a universal cover.*

Definition 6.2. For $\pi : \tilde{X} \to X$ a (universal) covering map, a deck transformation is a homeo $f : \tilde{X} \to \tilde{X}$ s.t. $\pi f = f$. These form a group $\mathrm{Aut}_\pi(\tilde{X})$.

Suppose G acts on Y by homeomorphisms

Definition 6.3. Recall faithful, free. An action is called a covering space action if every $y \in Y$ has an open nbd U_y s.t. $g_1 U_y \cap g_2 U_y = \varnothing$ for all $g_1 \neq g_2$.

The quotient map is cts, surj, and open.

Theorem 6.4. *If $\pi : \tilde{X} \to X$ is a covering map with $\tilde{X}$ conn. and Hausdorff, then the action of $\mathrm{Aut}_\pi(X)$ on $\tilde{X}$ is a covering space action.*

Proof. For $y \in \tilde{X}$, pick nbd N of $\pi(y)$ which is evenly covered. Let U_y be the component of $\pi^{-1}(N)$ containing y. If $g(U_y) \cap U_y \neq \varnothing$, then find $p_1, p_2 \in U_y$ s.t. $gp_1 = p_2$, but π is inj on U_y, so $p_1 = p_2$, so g has a fixed point, so $g = e$. □

Theorem 6.5. *Y conn, loc. path conn, Hausdorff with a G acting on Y faithfully.*

(i) $q : Y \to Y/G$ is a covering map iff the action of G is a covering space action

(ii) If G is a covering space action, then $G = \mathrm{Aut}_q(Y)$

[Warning: Y/G needs not be Hausdorff, cf. ES3.]

Proof. (ii) G acts transitively on the fiber $q^{-1}(z)$. For $y \in q^{-1}(z)$ and $\gamma \in \mathrm{Aut}_q(Y)$, have $q\gamma(y) = q(y)$, so there exists $g \in G$ with $\gamma(y) = g(y)$, so $\gamma = g$. $\square$

Corollary 6.6. *If $\pi : \tilde{X} \to X$ is a universal cover, then $\tilde{X}/\mathrm{Aut}_\pi(\tilde{X}) \cong X$.*

Proof. Define $h : \tilde{X}/\mathrm{Aut}_\pi(\tilde{X}) \to X, [y] \mapsto \pi(y)$. Then $\pi = hq$. h is open surjective, so check injectivity, i.e., $\pi y = \pi z \implies \exists \gamma \in \mathrm{Aut}_\pi(\tilde{X})$ with $\gamma y = z$, i.e., the action on fiber is transitive. This is true as $\tilde{X}$ is simply connected. $\square$

Proposition 6.7. *If X is a metric space and G acts by isometries and the action is a covering space action, then X/G is Hausdorff.*

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If R is a Riemann surface and $\pi : \tilde{R} \to R$ a universal cover, then $\tilde{R}$ is a Riemann surface and π is analytic. If $\pi' : \tilde{R}' \to R$ another universal cover with $\pi = \pi' H$. Have H, H^{-1} analytic.

Proposition 6.8. *Any $g \in \mathrm{Aut}_\pi(\tilde{R})$ is analytic*

Proof. Check directly. $\square$

So $\mathrm{Aut}_\pi(\tilde{R})$ is a group of analytic automorphisms of $\tilde{R}$ giving a covering space action. Conversely, if $\tilde{R}$ is simply conn Riemann surface with charts ρ_i and G a group of analytic automorphisms of $\tilde{R}$ which gives a covering space action, then $q : \tilde{R} \to \tilde{R}/G$ is a covering map. If $\tilde{R}/G$ is Hausdorff, then it's a connected surface with charts of the form $\rho_i(q|_U)^{-1}$. It's a Riemann surface since transition functions are of the form $\rho_i(q|_{U_y})^{-1}(q|_{U_z})\rho_j^{-1}$ (the map in the middle is given by some element $g \in G$) Can check that R is analytically isomorphic to $\tilde{R}/G$. So Riemann surfaces are exactly simply connected Riemann surfaces/covering space actions of analytic automorphisms.

Theorem 6.9 (Uniformization)**.** *Up to analytic isomorphism, the simply connected Riemann surfaces are*

(i) $\mathbb{C}$

(ii) $\mathbb{C}_\infty$

(iii) $\mathbb{H}$ *(upper half plane) or* $\mathbb{D}$ *(unit disk)*

These three are distinct.
Automorphisms

(i) Automorphism groups of $\mathbb{C}_\infty$ is given by $\mathcal{M}$ (Mobius transformations). All have fixed pts

(ii) $\mathrm{Aut}(\mathbb{C}) = \{az + b : a \neq 0\}$. Have fixed pt if $a \neq 1$. So look at subgroups of $\{z \mapsto z + b\} \cong \mathbb{C}$. For this to give a covering space action, consider 5.3, so we get $\mathbb{C}, \mathbb{C} \setminus \{0\}$ or various tori $\mathbb{C}/L$.

Corollary 6.10. *Any Riemann surface R other than the above is given by $\mathbb{H}/T$ for $T \leq \mathrm{Aut}(\mathbb{H}) = \{\frac{az+b}{cz+d} : a, b, c, d \in \mathbb{R},\ ad - bc = 1\}$ with a covering space action.*

Proof. Schwarz lemma (cf. CA) $\implies \mathrm{Aut}(\mathbb{H})$ is this group. It acts on $\mathbb{H}$ by isometries for $d_{\mathbb{H}}(z, w) = 2\tanh^{-1}|\frac{z-w}{z-\bar{w}}|$ (cf. Geo). $\square$

Corollary 6.11 (Riemann mapping thm)**.** *If U is a simply connected proper open subset of $\mathbb{C}$, then U is conformally equivalent to $\mathbb{D}$.*

Proof. U is its own universal cover, so $U = U/\{\mathrm{id}\}$ so $U \cong \mathbb{C}$ or $\mathbb{D}$. Can't be $\mathbb{C}$ since any inj holo'c map $\mathbb{C} \to \mathbb{C}$ is surj. $\square$

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